Stochastic approaches to wave turbulence

Daniel Schubring

October 27, 2023

Rosenhaus, D.S, Shuvo, Smolkin (arXiv:2308.00740), D.S (arXiv:2309.08484)

Equilibrium and non-equilibrium physics

- Leaving familiar territory...
  - Thermal equilibrium
  - Local equilibrium and hydrodynamics
  - Turbulent stationary states
- Low orders of perturbation theory: Kolmogorov-Zakharov (KZ) spectrum
- Can it be extended to higher orders? Non-perturbatively?

Rosenhaus, D.S, Shuvo, Smolkin (arXiv:2308.00740), D.S (arXiv:2309.08484)

Richardson cascade

"Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls, and so on to viscosity." (1922)

- 'Inertial range' in between forcing and dissipation
- Stationary state parametrized by flux ϵ of conserved energy
- Kolmogorov 1941. Dimensional analysis with ϵ, ρ

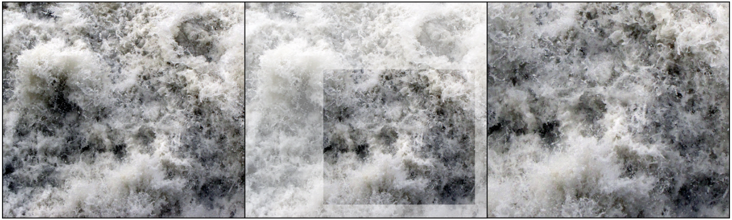


Image: Steven Mathey. Neckar river, Heidelberg. (arXiv:1410.8712)

Examples of wave turbulence

- Will focus on classical field theories (...but Schwinger-Keldysh)
- Examples of wave turbulence
  - Gravity and capillary waves (fluids)
  - Gravitational waves (cosmology)
  - Classical Yang-Mills (quark gluon plasma)
- Weak wave turbulence paradigm
  - Express in terms of classical a, a^* fields
  - Truncate interaction to low orders of non-linearity
  - Find and solve wave kinetic equation

Galtier, Nazarenko 2017 (arXiv:1703.09069). Mehtar-Tani 2016 (arXiv:1611.01527)

Example: Capillary waves

- η is height of fluid, ϕ is velocity potential $v = \nabla\phi$.
- Form complex 'ladder operator' field

$$a_k = \sqrt{\frac{\sigma k^2}{2\omega_k}}\eta_k - i\sqrt{\frac{k}{2\omega_k}}\phi_k.$$

- Truncated 'three-wave' Hamiltonian

$$H = \sum_k \omega_k a_k^* a_k + \frac{1}{2}\sum_{ijl}\left(\lambda_{l;ij} a_l^* a_i a_j + \text{c.c}\right) + \mathcal{O}\left(a^4\right)$$

$$\omega_k \sim k^{3/2}, \qquad \lambda_{kl;ki,kj} \sim k^{9/4}\,\tilde{\lambda}_{l;ij}$$

Pushkarev, Zakharov. Phys. Rev. Lett. 76, 3320 (1996)

Wave kinetic equation

$$H = \sum_k \omega_k a_k^* a_k + \frac{1}{2} \sum_{ijl} \left(\lambda_{l;ij} a_l^* a_i a_j + \text{c.c} \right)$$

- Find time derivative of $a_r^* a_r$,

$$\{a_r^* a_r, H\} = \sum_{ij} \operatorname{Im} \left[\lambda_{r;ij} a_r^* a_i a_j - 2\lambda_{i;jr} a_i^* a_j a_r \right].$$

- Take expectation value (how?), $n_r \equiv \langle a_r^* a_r \rangle$.

$$\frac{dn_r}{dt} = \sum_{ij} \operatorname{Im} \left[\lambda_{r;ij} \langle a_r^* a_i a_j \rangle - 2\lambda_{i;jr} \langle a_i^* a_j a_r \rangle \right]. \qquad \text{(wave kin. eq.)}$$

- $\frac{dn_r}{dt} = 0$ for either thermal equilibrium or KZ.

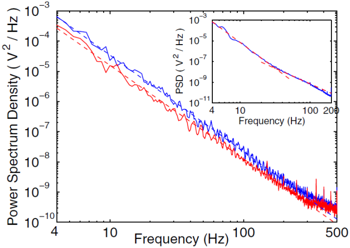
$$n_k = T\omega_k^{-1} \qquad \text{or} \qquad n_k \sim \omega_k^{-17/6}$$

Capillary waves: experimental evidence

- Going back to definition of a_k,

$$\langle|\eta_k|^2\rangle = \frac{\omega_k}{\sigma k^2} n_k \sim \omega_k^{-19/6}.$$

- Forced turbulence in ethanol tank on Airbus A310 Zero G
- Red: random forcing, slope -3.1
- Blue: periodic forcing, slope -3.2



C. Falcón et al 2009 EPL 86 14002 (arXiv:0708.1446)

How are expectation values calculated?

- The KZ solution involved solving kinetic equation

$$\frac{dn_r}{dt} = \sum_{ij} \mathsf{Im}\left[\lambda_{r;ij}\left\langle a_r^* a_i a_j\right\rangle - 2\lambda_{i;jr}\left\langle a_i^* a_j a_r\right\rangle\right] = 0.$$

- Need to express $\langle a_r^* a_i a_j \rangle$ in terms of the n_r.
- Expectation values are calculated with respect to some $\rho(a,\, a^*)$.
- A zeroth order ρ_0 is taken to be Gaussian

$$\rho_0 = e^{-\sum_k \frac{1}{n_k} a_k^* a_k}.$$

- Wyld's method: ρ_0 applies to free fields $a^{(0)}$. Time dependent.

Liouville Hamiltonian

- We really want statistics in the stationary state
- ρ obeys Liouville equation in inertial range $\dot{\rho} = -\{\rho, H\}$.
- Stationary states are zero eigenvectors of *Liouville Hamiltonian*

$$\hat{H}_L \rho \equiv \{\rho, H\} = 0.$$

- Analogies to quantum mechanics (Prigogine)
- Can find ρ perturbatively from zeroth-order ρ_0
- But $\hat{H}_L$ is rather pathological

Gurarie 1995 (hep-th/9405077), Rosenhaus, Smolkin 2022 (2212.02555).

Langevin equation for thermal equilibrium

- Helpful to consider thermal equilibrium case
- Need to modify dynamics such that ρ evolves to $\rho_B = e^{-\frac{1}{T}H}$.
- This can be done by adding dissipation and random forcing

$$\dot{a} = -\frac{\gamma}{\omega}\frac{\partial H}{\partial a^*} + f - i\frac{\partial H}{\partial a^*},$$

$$\langle f(t) f(t_0)\rangle = 2\frac{\gamma}{\omega} T\,\delta(t-t_0).$$

- Dissipation drives to local minima $\frac{\partial H}{\partial a} = \frac{\partial H}{\partial a^*} = 0$.
- Random forcing involves temperature T
- Final ‘inertial’ term doesn’t affect late time distribution.
- In the end can take $\gamma \to 0$.

Langevin equation for non-equilibrium states

- Now simply drive different modes with distinct temperatures

$$\dot{a}_k = -\frac{\gamma}{\omega}\frac{\partial H}{\partial a_k^*} + f_k - i\frac{\partial H}{\partial a_k^*},$$
$$\langle f_k(t) f_l(0)\rangle = 2\frac{\gamma_k}{\omega_k} T_k\, \delta(t)\delta_{kl}.$$

- For non-interacting H_0, the late time distribution is just

$$\rho_0 = \prod_k e^{-\frac{\omega_k}{T_k} a_k^* a_k} = e^{-\sum_k \frac{1}{n_k} a_k^* a_k}.$$

- The ρ corresponding to the full H will have non-Gaussian corrections
- Can again consider $\gamma_k \to 0$ limit

The big picture

- We want to consider the turbulent KZ state at higher order
- The Langevin equation determines non-perturbative stationary states
- Expectation values like $\langle a_r^* a_i a_j \rangle$ may be calculated
- These determine the RHS of kinetic equation, and thus the KZ state
- Three equivalent methods:

$$\text{Langevin equation} \leftrightarrow \text{Fokker-Planck Hamiltonian} \leftrightarrow \text{Path integral}$$

Fokker-Planck equation

- Langevin equation determines $a_f(t; a_0)$ given initial conditions a_0
- Averaging over f determines a distribution ρ

$$\rho(a, t; a_0) = \langle \delta(a - a_f(t; a_0)) \rangle$$

- The Langevin equation for a_f implies an equation for ρ

$$\dot{\rho} = -\left(\hat{H}_L + \hat{H}_\gamma\right)\rho$$

  - The Hamiltonian part of the Langevin equation leads to $\hat{H}_L$,

$$\hat{H}_L \rho \equiv \{\rho, H\} = -i \sum_k \left(\frac{\partial \rho}{\partial a_k} \frac{\partial H}{\partial a_k^*} - \frac{\partial H}{\partial a_k} \frac{\partial \rho}{\partial a_k^*} \right).$$

  - The dissipation and forcing parts lead to

$$\hat{H}_\gamma \rho = -\sum_k \frac{\gamma_k}{\omega_k} \frac{\partial}{\partial a_k^*} \left[\frac{\partial H}{\partial a_k} \rho + T_k \frac{\partial \rho}{\partial a_k} \right] + \text{c.c.}$$

Linear and non-linear dissipation

$$\hat{H}_L\rho \equiv \{\rho, H\}, \qquad \hat{H}_\gamma\rho = -\sum_k \frac{\gamma_k}{\omega_k}\frac{\partial}{\partial a_k^*}\left[\frac{\partial H}{\partial a_k}\rho + T_k\frac{\partial \rho}{\partial a_k}\right] + \text{c.c.}$$

- Since both $\hat{H}_L$ and $\hat{H}_\gamma$ are total derivatives, probability is conserved
- If $T_k = T$, $\rho_B = e^{-\frac{H}{T}}$ satisfies $\left(\hat{H}_L + \hat{H}_\gamma\right)\rho_B = 0$.
- What if we linearize the dissipation term?

$$-\frac{\gamma}{\omega}\frac{\partial H}{\partial a^*} \to -\frac{\gamma}{\omega}\frac{\partial H_0}{\partial a^*} = -\gamma a$$

- ρ_B is no longer stationary state, but $\hat{H}_\gamma$ has no interaction dependence

Perturbation theory

- Goal is to calculate expectation values like $\langle a_r^* a_i a_j \rangle$
- Looking for stationary ρ, i.e. $\hat{H}\rho = 0$.
- Split up $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{V}\psi = \{\psi, V\}$
- Geometric series for ρ in terms of ρ_0,

$$\rho = \rho_0 - \hat{H}_0^{-1}\hat{V}\rho \quad \implies \quad \rho = \left(1 - \hat{H}_0^{-1}\hat{V} + \left(-\hat{H}_0^{-1}\hat{V}\right)^2 + \dots\right)\rho_0$$

- We need to solve eigenvalue problem for ρ^i, E_0^i

$$\hat{H}_0\rho^i = E_0^i\rho^i$$

Diagonalizing $\hat{H}_0$

- We want to solve $\hat{H}_0\rho = E_0\rho$
- Helpful to introduce rescaled action-angle variables, x, α

$$a = \sqrt{nx}e^{-i\alpha}$$

$$\left[\omega\partial_\alpha - 2\gamma\left(x\partial_x^2 + (x+1)\partial_x + \frac{1}{4x}\partial_\alpha^2 + 1\right)\right]\rho(x,\alpha) = E_0\rho(x,\alpha).$$

- Take ansatz

$$\rho_{\kappa,\nu}(x,\alpha) = \sqrt{x}^{|\nu|}e^{i\nu\alpha}\psi_{\kappa,\nu}(x)e^{-x}, \qquad E_0 = 2\gamma\kappa + \gamma|\nu| + i\nu\omega.$$

- Reduces to associated Laguerre equation

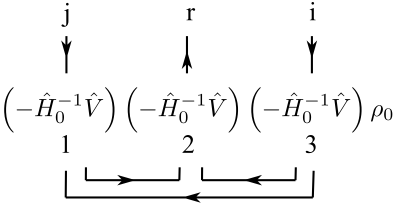
$$x\psi''_{\kappa,\nu} + (1+|\nu|-x)\psi'_{\kappa,\nu} + \kappa\psi_{\kappa,\nu} = 0.$$

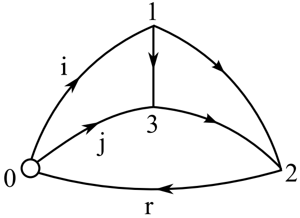
Example of a correction

- Want to calculate $\langle a_r a_i^* a_j^* \rangle^{(3)}$,

$$\langle a_r a_i^* a_j^* \rangle^{(3)} = \int dada^* \; a_r a_i^* a_j^* \left(-\hat{H}_0^{-1}\hat{V}\right)^3 \rho_0$$

- Recall $V = \frac{1}{2}\sum_{kij} \lambda_{k;ij} a_k^* a_i a_j +$ c.c.
- Integral vanishes unless each a paired with a^* of same mode
- Can construct time-ordered diagrams $t_0 < t_1 < t_2 < t_3$



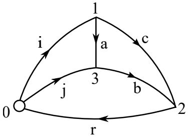


MSR path integral

- These same diagrams arise from a path integral approach
- Martin, Siggia, Rose 1973, Janssen 1976, de Dominicis 1976
- Comments on derivation
  - Directly from Fokker-Planck $\hat{H}$
  - From Langevin equation. Similar to Faddeev-Popov trick
  - Jacobian and non-linear dissipation
- Fields in path integral involve time t or frequency z
  - $Z = \int \mathcal{D}a\mathcal{D}a^* e^{-\int \frac{dz}{2\pi} \frac{1}{2\gamma n} \left|(-i(z-\omega)+\gamma)a+i\frac{\partial V}{\partial a^*}\right|^2}$
  - Free propagator $\int \frac{dz}{2\pi} \frac{2\gamma n\, e^{-izt}}{(z-\omega)^2+\gamma^2} = n\, e^{-i\omega t} e^{-\gamma|t|}$.
- But we only need equal time expectation values

Application to wave kinetic equation: Rosenhaus, Smolkin 2022 (2203.08168)

Old-fashioned perturbation theory



- Example: propagator j from t_0 to t_3: $n_j\, e^{-i\omega_j(t_3-t_0)} e^{-\gamma_j|t_3-t_0|}$
- Choosing a time ordering fixes absolute value sign
- Integrating over t_3 produces same frequency denominator as $\hat{H}_0^{-1}$

$$\int_{t_2}^{\infty} dt_3 e^{-(i(\omega_j+\omega_a-\omega_b)+\gamma_a+\gamma_b+\gamma_c)t_3} = \frac{e^{-(i(\omega_j+\omega_a-\omega_b)+\gamma_a+\gamma_b+\gamma_c)t_2}}{i(\omega_j+\omega_a-\omega_b)+\gamma_a+\gamma_b+\gamma_c}$$

- Then integrate over t_2 and t_1 for more denominators
- Algorithm for general diagrams in terms of simple rules

Rosenhaus, DS, Shuvo, Smolkin 2023 (2308.00740)

Coming back to turbulence

- Now we can calculate high-order correlation functions in the state ρ
- But ρ involves a distinct temperature T_k for each mode. Too much freedom?
- If we can set $\gamma \to 0$, this would imply independent conserved quantities

$$\dot{\rho} = -\left(\hat{H}_L + \hat{H}_\gamma\right)\rho \to -\{\rho, H\} = 0.$$

- There must be problems with the $\gamma \to 0$ limit for most choices of T_k
- The KZ state avoids some pathologies in this limit

Ehrenfest theorem

- Take time derivative of expectation value of $G(a, a^*)$

$$\frac{d}{dt}\langle G\rangle = -\int dada^*\, G\left(\hat{H}_L + \hat{H}_\gamma\right)\rho$$
$$= \langle\{G, H\}\rangle - \int dada^*\, G\, \hat{H}_\gamma \rho.$$

- For a stationary state (linear dissipation)

$$\langle\{G, H\}\rangle = \sum_k \gamma_k \left\langle a_k \frac{\partial G}{\partial a_k} + a_k^* \frac{\partial G}{\partial a_k^*} - 2n_k \frac{\partial^2 G}{\partial a_k \partial a_k^*}\right\rangle.$$

- Using $G = a_r^* a_r$ will give wave kinetic equation on LHS

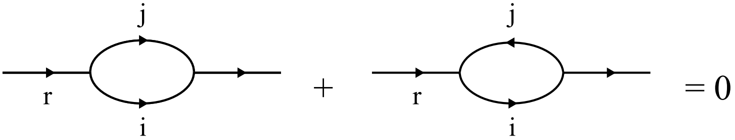
Wave kinetic equation

- So there is alternate formulation of kinetic equation

$$\lim_{\gamma\to 0}\left\langle \frac{d}{dt} a_r^* a_r \right\rangle = \lim_{\gamma\to 0} 2\gamma_r \left[\langle a_r^* a_r\rangle - n_r\right]$$

- $\lim\limits_{\gamma\to 0}\langle a_r^* a_r\rangle$ diverges except for KZ state
- Kinetic equation interpreted diagramatically

$$\sum_{ij} \mathsf{Im}\left[\lambda_{r;ij}\langle a_r^* a_i a_j\rangle - 2\lambda_{i;jr}\langle a_i^* a_j a_r\rangle\right] = 0.$$



Higher-order correlations?

- Does the KZ state avoid all divergences as $\gamma \to 0$?
- Choose $G = a_1^* a_1 a_2^* a_2$ in the Ehrenfest equation
- Similarly

$$\lim_{\gamma \to 0} \left\langle \frac{d}{dt} a_1^* a_1 a_2^* a_2 \right\rangle = \lim_{\gamma \to 0} 2 \left(\gamma_1 + \gamma_2 \right) \left\langle a_1^* a_1 a_2^* a_2 \right\rangle_c \neq 0?$$

- Divergence avoided for thermal equilibrium, but not KZ?

Conclusion

- Summary
  - Treating weak wave turbulence in terms of stationary states by introducing auxilliary forcing and dissipation
  - Efficient method to calculate correlation functions and corrections to the kinetic equation
  - A non-perturbative understanding of the wave kinetic equation
- Remaining theoretical problems
  - Solution to the higher-order kinetic equation?
  - Better connection to time-dependent methods
  - Understanding of γ divergences in higher correlation functions
- Some future directions
  - Large N model
  - Quantum mechanical turbulence

Shavit, Falkovich 2020 (1911.12670)

Thank you!

- Based on papers (2023):
  - Rosenhaus, D.S., Shuvo, Smolkin, *Loop diagrams in the kinetic theory of waves* (arXiv:2308.00740)
  - D.S., *Fokker-Planck approach to wave turbulence* (arXiv:2309.08484)
- Capillary waves example:
  - Theory: Pushkarev, Zakharov. Phys. Rev. Lett. 76, 3320 (1996)
  - Experiment: C. Falcón et al. EPL 86 14002 (2009) (arXiv:0708.1446)

Constructing the path integral

- Solve for $a_E(t)$ given $f, a(t_0), E$,

$$\dot{a} + \left(\frac{\gamma}{\omega} + i\right)\frac{\partial H}{\partial a^*} - f = E.$$

- The value of a function $G(a, a^*)$ 'on-shell' may be calculated

$$\begin{aligned} G(a_0, a_0^*) &= \int \mathcal{D}E\mathcal{D}E^* \delta(E)\delta(E^*)G(a_E, a_E^*) \\ &= \int \mathcal{D}a\mathcal{D}a^*\mathcal{D}\eta\mathcal{D}\eta^* \frac{\partial\left(E,\, E^*\right)}{\partial\left(a,\, a^*\right)} e^{i\int dt(\eta E^* + \eta^* E)} G(a, a^*) \end{aligned}$$

- Now average over Gaussian statistics of forcing function f

$$\langle G(a, a^*)\rangle = \int \mathcal{D}f\mathcal{D}f^* e^{-\int dt \frac{|f|^2}{2\gamma n}} \ldots$$