Stochastic approaches to wave turbulence

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March 14, 2024

Rosenhaus, D.S, Shuvo, Smolkin (arXiv:2308.00740), D.S (arXiv:2309.08484)

The main point

- Weak wave turbulence paradigm
  - Find wave kinetic equation at leading order in a small parameter
  - Study turbulent (KZ) stationary states
- Can it be extended to higher orders? Non-perturbatively?
- Our approach involves auxilliary stochastic forcing, dissipation
  - Non-equilibrium stationary state from ‘equilibrium’ methods
  - Simple(ish) algorithm to calculate corrections to collision integral
  - Can derive kinetic equation in a large N model

Rosenhaus, D.S, Shuvo, Smolkin (arXiv:2308.00740), D.S (arXiv:2309.08484), Large N: Rosenhaus, D.S (to appear)

What is the system?

- Consider classical field theories in terms of field a_k

$$H = \sum_k \omega_k a_k^* a_k + \sum_{ijkl} \lambda_{ij;kl}\, a_i^* a_j^* a_k a_l$$

- Depending on choice of ω, λ,
  - Non-linear Schrödinger
  - Gravity waves
  - Capillary waves?

Image: Will Dendis

Deriving kinetic equation

$$H = \sum_k \omega_k a_k^* a_k + \sum_{ijkl} \lambda_{ij;kl}\, a_i^* a_j^* a_k a_l$$

- Equations of motion for 'wave action' $a_r^* a_r$

$$\frac{d}{dt} a_r^* a_r = \{a_k^* a_k, H\} = 4 \sum_{jkl} \mathsf{Im}\left(\lambda_{rj;kl} a_r^* a_j^* a_k a_l\right)$$

- Take an ensemble average, $\langle a_r^* a_r \rangle \equiv n_r$
- Closure: $\langle a_r^* a_j^* a_k a_l \rangle$ can be expressed in terms of n (How?)

Weak wave turbulence

- First order approximation to $\langle a_r^* a_j^* a_k a_l \rangle$

$$\frac{d}{dt} n_r = 4 \sum_{jkl} \mathrm{Im}\left(\lambda_{rj;kl} \langle a_r^* a_j^* a_k a_l \rangle\right)$$
$$\propto \sum_{jkl} \left|\lambda_{rj;kl}\right|^2 \left(\prod n\right) \left(\frac{1}{n_r} + \frac{1}{n_j} - \frac{1}{n_k} - \frac{1}{n_l}\right) \delta\left(\sum \omega\right)$$

- This has equilibrium solutions $n_i = T / \left(\omega_i - \mu\right)$
- Also non-trivial KZ solutions: $n(k_i) \propto |k_i|^{-\gamma}$
- Do these solutions exist at higher order?

Phase space picture

- Turbulence as stationary ensemble $\rho(a, a^*)$ on phase space
- Can calculate expectation values like $\langle a_r^* a_j^* a_k a_l \rangle$

- By Liouville equation,

$$\frac{d\rho}{dt} = -\{\rho, H\} = 0.$$

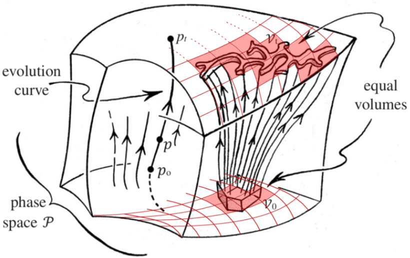


Image: Javier Martin-Garcia (arXiv:2209.04632)

A formal solution

- Liouville 'Hamiltonian': $\hat{H}\rho = \{\rho, H\}$
- Looking for zero eigenfunctions $\hat{H}\rho = 0$
- Find solution ρ_0 to unperturbed Hamiltonian

$$\hat{H}_0\rho_0 = -\left\{\rho_0, \sum_k \omega_k a_k^* a_k\right\} = 0$$

- A lot of solutions. Pick $\rho_0 \propto e^{-\sum_k \frac{1}{n_k} a_k^* a_k}$
- Full solution $\left(\hat{H}_0 + \hat{V}\right)\rho = 0$, if ρ satisfies

$$\rho = \rho_0 - \hat{H}_0^{-1}\hat{V}\rho$$

- Closure: ρ depends on wave action n_k in ρ_0

A perturbative solution

$$\rho = \left(1 - \hat{H}_0^{-1}\hat{V} + \left(-\hat{H}_0^{-1}\hat{V}\right)^2 + \dots\right)\rho_0$$

- First order: $\hat{V}\rho_0 = i\sum\left(\dots\right) a_i^* a_j^* a_k a_l \rho_0$
- But 'problem of small denominators':

$$\hat{H}_0 a_i^* a_j^* a_k a_l \rho_0 = i\left(\omega_i + \omega_j - \omega_k - \omega_l\right) a_i^* a_j^* a_k a_l \rho_0$$

- How to take $\hat{H}_0^{-1}\hat{V}\rho_0$ if there are resonances?
- Ad-hoc solution: Regulate $\hat{H}_0{}^{-1} \to \left(\hat{H}_0 - \epsilon\right)^{-1}$
- Finds kinetic equation! But desire more well-behaved $\hat{H}_0$

Gurarie 1995 (hep-th/9405077), Rosenhaus, Smolkin 2022 (2212.02555).

Introducing forcing and dissipation

- Goal: Regulate Liouville Hamiltonian
- Modify classical equations of motion (one d.o.f.)

$$\dot{a} = -i\frac{\partial H}{\partial a^*} \quad \to \quad \dot{a} = -i\frac{\partial H}{\partial a^*} - \frac{\gamma}{\omega}\frac{\partial H}{\partial a^*} + f.$$

- Dissipation term drives to local minima of H
- Random forcing involves temperature T

$$\langle f(t) f(t_0)\rangle = 2\frac{\gamma}{\omega}T\,\delta(t - t_0).$$

- Late time distribution is just $\rho \propto e^{-\frac{1}{T}H}$.
- In the end can take $\gamma \to 0$ limit

Many degrees of freedom

- Now simply drive different modes with distinct temperatures

$$\dot{a}_k = -i\frac{\partial H}{\partial a_k^*} - \frac{\gamma}{\omega}\frac{\partial H}{\partial a_k^*} + f_k,$$
$$\langle f_k(t) f_l(0)\rangle = 2\frac{\gamma_k}{\omega_k}T_k\,\delta(t)\delta_{kl}.$$

- For non-interacting H_0, the late time distribution is just

$$\rho_0 = \prod_k e^{-\frac{\omega_k}{T_k}a_k^* a_k} = e^{-\sum_k \frac{1}{n_k}a_k^* a_k}.$$

- The ρ corresponding to the full H will have non-Gaussian corrections
- Can again consider $\gamma_k \to 0$ limit

The big picture

- Classical equations $\to$ Langevin equations. Parameters γ_k, n_k
- Liouville Hamiltonian $\to$ Fokker-Planck Hamiltonian
- Solve perturbatively for non-equilibrium stationary state ρ
- Expectation values like $\langle a_r^* a_j^* a_k a_l \rangle$ may be calculated
- These determine the collision integral of kinetic equation
- Finally take the limit $\gamma_k \to 0$ (if possible)
- Equivalent approach: Martin-Siggia-Rose path integral
- (MSR approach related to Schwinger-Keldysh)

Introducing Fokker-Planck equation

- Completely analogous to Liouville equation
- Langevin equation implies time dependence for phase space functions

$$\frac{d\rho}{dt} = -\left(\hat{H}_L + \hat{H}_\gamma\right)\rho$$

- $\hat{H}_L\rho = \{\rho, H\}$ is just the Liouville Hamiltonian, as before
- Dissipation and forcing leads to

$$\hat{H}_\gamma\rho = -\sum_k \frac{\gamma_k}{\omega_k}\frac{\partial}{\partial a_k^*}\left[\frac{\partial H}{\partial a_k}\rho + T_k\frac{\partial \rho}{\partial a_k}\right] + \text{c.c.}$$

More on the Fokker-Planck equation

$$\hat{H}_L \rho \equiv \{\rho, H\}, \qquad \hat{H}_\gamma \rho = -\sum_k \frac{\gamma_k}{\omega_k} \frac{\partial}{\partial a_k^*} \left[\frac{\partial H}{\partial a_k} \rho + T_k \frac{\partial \rho}{\partial a_k} \right] + \text{c.c.}$$

- If $T_k = T$, $\left(\hat{H}_L + \hat{H}_\gamma \right) e^{-\frac{H}{T}} = 0$.
- What if we linearize the dissipation term? $-\frac{\gamma}{\omega} \frac{\partial H}{\partial a^*} \to -\frac{\gamma}{\omega} \frac{\partial H_0}{\partial a^*} = -\gamma a$
  - Good: $\hat{H}_\gamma$ has no interaction dependence, simpler perturbation theory
  - Bad: Thermal equilibrium is not a stationary state at finite γ
  - Since we are ultimately taking $\gamma \to 0$, perhaps this is fine
- In either case, the problem of diagonalizing $\hat{H}_0$ is much clearer
  - Can be reduced to the associated Laguerre equation
  - A unique zero eigenvector
  - The role of the ad-hoc ϵ is played by γ

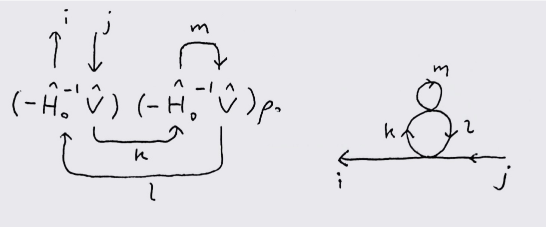
D.S. 2023 (arXiv:2309.08484)

Diagrams from perturbative corrections

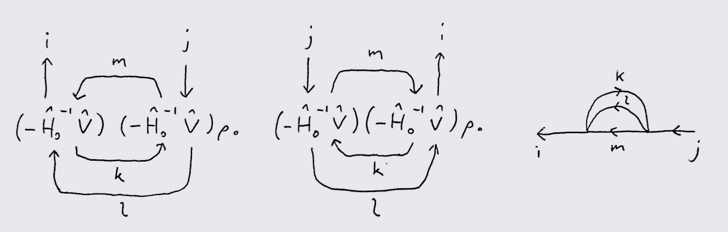
- Example: Want to calculate $\langle a_i a_j^* \rangle^{(2)}$,

$$\langle a_i a_j^* \rangle^{(2)} = \int dada^* \; a_i a_j^* \left(-\hat{H}_0^{-1}\hat{V}\right)^2 \rho_0$$

- Recall $V = \sum_{ijkl} \lambda_{ij;kl} a_i^* a_j^* a_k a_l$
- Integral vanishes unless each a paired with a^* of same mode
- Example of pairing



Time orderings of vertices



- A contribution really involves an ordering of the vertices
- These diagrams also arise in the MSR path integral
- Each line is associated to a propagator $ne^{-i\omega t}e^{-\gamma|t|}$
- The ordering of vertices is just time-ordering
- There are simple rules for evaluating time-ordered diagrams

Rosenhaus, D.S, Shuvo, Smolkin, 2023 (arXiv:2308.00740)

Coming back to turbulence

- Now we can calculate high-order correlation functions in the state ρ
- But ρ involves a distinct T_k or n_k for each mode. Too much freedom?
- If we can set $\gamma \to 0$, this would imply independent conserved quantities
$$\dot{\rho} = -\left(\hat{H}_L + \hat{H}_\gamma\right)\rho \to -\{\rho, H\} = 0.$$
- There must be problems with the $\gamma \to 0$ limit for most choices of n_k
- The KZ state avoids some pathologies in this limit

Ehrenfest theorem

- Take time derivative of expectation value of $G(a, a^*)$

$$\begin{aligned}\frac{d}{dt}\langle G\rangle &= -\int dada^*\, G\left(\hat{H}_L + \hat{H}_\gamma\right)\rho \\ &= \langle\{G, H\}\rangle - \int dada^*\, G\, \hat{H}_\gamma \rho.\end{aligned}$$

- For a stationary state (linear dissipation)

$$\langle\{G, H\}\rangle = \sum_k \gamma_k \left\langle a_k \frac{\partial G}{\partial a_k} + a_k^* \frac{\partial G}{\partial a_k^*} - 2n_k \frac{\partial^2 G}{\partial a_k \partial a_k^*}\right\rangle.$$

- Using $G = a_r^* a_r$ will give wave kinetic equation

D.S. 2023 (arXiv:2309.08484)

Back to wave kinetic equation

- This relates a 4-point expectation to a 2-point expectation

$$\begin{aligned}\langle\{a_r^* a_r, H\}\rangle &= 4\sum_{jkl} \operatorname{Im}\left(\lambda_{rj;kl}\langle a_r^* a_j^* a_k a_l\rangle\right)\\ &= 2\gamma_r\left[\langle a_r^* a_r\rangle - n_r\right]\end{aligned}$$

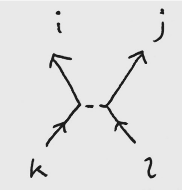
- In general $\langle a_r^* a_r\rangle \propto (2\gamma_r)^{-1}$
- The $\gamma \to 0$ limit does not exist for most choices of n_k
- Collision integral may be found from $\langle a_r^* a_r\rangle$

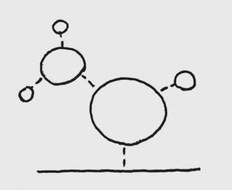
Large N model

- Can sum certain corrections to all orders
- Extend a_k to N component field $\vec{a}_k$

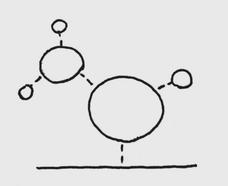
$$H = \sum_k \omega_k \vec{a}^*_k \cdot \vec{a}_k + \frac{1}{N} \sum_{ijkl} \lambda_{ij;kl} \left(\vec{a}^*_i \cdot \vec{a}_k\right) \left(\vec{a}^*_j \cdot \vec{a}_l\right)$$

- Can order diagrams in powers of $1/N$
  - Each vertex leads to a factor of $1/N$
  - Each loop leads to a factor of N





Cactus diagrams



- ‘Cacti’ have no cost in $1/N$
- This amounts to a renormalization of ω, n, γ

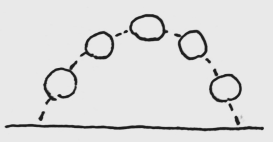
$$\tilde{\omega}_k = \omega_k + 2\sum_l \lambda_{kl;kl}\tilde{n}_l,$$

- The renormalization of n is akin to ‘saddle point’ or ‘gap’ equations

$$\tilde{n}_k = \frac{n_k}{1 + \frac{2}{\omega_k}\sum_l \lambda_{kl;kl}\tilde{n}_l}$$

- For linear dissipation the cacti simplify and n is not renormalized

Bubble chains



- A bubble chain arc contributes at order $1/N$
- This amounts to a renormalization of $\lambda_{ij;kl}$

$$\tilde{\lambda} = \frac{\lambda}{1 - \lambda\mathcal{L}}$$

- $\mathcal{L}$ is an integral derived from a single bubble $\mathcal{L} \propto \int d^d k \frac{\Delta n}{\Delta\omega + i\gamma}$
- Collision integral takes same form as before

$$\frac{d}{dt}\tilde{n}_r \propto \frac{1}{N}\sum_{jkl}\left|\tilde{\lambda}_{rj;kl}\right|^2 \left(\prod \tilde{n}\right)\left(\frac{1}{\tilde{n}_r} + \frac{1}{\tilde{n}_j} - \frac{1}{\tilde{n}_k} - \frac{1}{\tilde{n}_l}\right)\delta\left(\sum \tilde{\omega}\right)$$

A change in scaling?

- So what is different?
- Some naive analyses:

$$\tilde{\lambda} = \frac{\lambda}{1 - \lambda\mathcal{L}}, \qquad \lambda\mathcal{L} \propto \lambda \int d^d k \frac{\Delta n}{\Delta\omega}$$

  - For $\tilde{\omega}, \tilde{\lambda}$ to have same scaling as unrenormalized ω, λ

$$\tilde{\omega} \sim |k|^{\alpha}, \quad \tilde{\lambda} \sim |k|^{\beta}, \quad \tilde{n} \propto |k|^{-\gamma}$$

$$\gamma = d + \beta - \alpha$$

  - In the 'non-perturbative' limit where $\lambda\mathcal{L} \gg 1$, $\tilde{\lambda} \approx \lambda/(\lambda\mathcal{L})$

$$\beta \to \gamma + \alpha - d$$

Conclusion

- Summary
  - Modified equations of motion by auxilliary forcing and dissipation
  - Efficient method to calculate correlation functions and corrections to the kinetic equation
  - Ehrenfest theorem: Non-perturbative relations between correlation functions
  - A kinetic equation for a large N theory
- Remaining puzzles
  - Interpretation of large N kinetic equation
  - Understanding of γ divergences in higher correlation functions

Thank you!

- Based on papers:
  - Rosenhaus, D.S., Shuvo, Smolkin, *Loop diagrams in the kinetic theory of waves* (arXiv:2308.00740)
  - D.S., *Fokker-Planck approach to wave turbulence* (arXiv:2309.08484)
  - Kinetic equation at large N: work in progress with V. Rosenhaus
- See also
  - Rosenhaus, Smolkin, *Feynman rules for forced wave turbulence* (arXiv:2203.08168)
  - Rosenhaus, Falkovich, *Interaction renormalization and validity of kinetic equations for turbulent states* (arXiv:2308.00033)

MSR path integral

- These same diagrams arise from a path integral approach
- Martin, Siggia, Rose 1973, Janssen 1976, de Dominicis 1976
- Comments on derivation
  - Directly from Fokker-Planck $\hat{H}$
  - From Langevin equation. Similar to Faddeev-Popov trick
  - Jacobian and non-linear dissipation
- Fields in path integral involve time t or frequency z
  - $Z = \int \mathcal{D}a\mathcal{D}a^* e^{-\int \frac{dz}{2\pi}\frac{1}{2\gamma n}\left|(-i(z-\omega)+\gamma)a+i\frac{\partial V}{\partial a^*}\right|^2}$
  - Free propagator $\int \frac{dz}{2\pi}\frac{2\gamma n\, e^{-izt}}{(z-\omega)^2+\gamma^2} = n\, e^{-i\omega t}e^{-\gamma|t|}$.
- But we only need equal time expectation values

Application to wave kinetic equation: Rosenhaus, Smolkin 2022 (2203.08168)

Constructing the path integral

- Solve for $a_E(t)$ given $f, a(t_0), E$,

$$\dot{a} + \left(\frac{\gamma}{\omega} + i\right)\frac{\partial H}{\partial a^*} - f = E.$$

- The value of a function $G(a, a^*)$ 'on-shell' may be calculated

$$\begin{aligned} G(a_0, a_0^*) &= \int \mathcal{D}E\mathcal{D}E^* \delta(E)\delta(E^*) G(a_E, a_E^*) \\ &= \int \mathcal{D}a\mathcal{D}a^*\mathcal{D}\eta\mathcal{D}\eta^* \frac{\partial\left(E,\, E^*\right)}{\partial\left(a,\, a^*\right)} e^{i\int dt(\eta E^* + \eta^* E)} G(a, a^*) \end{aligned}$$

- Now average over Gaussian statistics of forcing function f

$$\langle G(a, a^*)\rangle = \int \mathcal{D}f\mathcal{D}f^* e^{-\int dt \frac{|f|^2}{2\gamma n}} \dots$$

Diagonalizing $\hat{H}_0$

- We want to solve $\hat{H}_0\rho = E_0\rho$
- Helpful to introduce rescaled action-angle variables, x, α

$$a = \sqrt{nx}e^{-i\alpha}$$

$$\left[\omega\partial_\alpha - 2\gamma\left(x\partial_x^2 + (x+1)\partial_x + \frac{1}{4x}\partial_\alpha^2 + 1\right)\right]\rho(x,\alpha) = E_0\rho(x,\alpha).$$

- Take ansatz

$$\rho_{\kappa,\nu}(x,\alpha) = \sqrt{x}^{|\nu|}e^{i\nu\alpha}\psi_{\kappa,\nu}(x)e^{-x}, \qquad E_0 = 2\gamma\kappa + \gamma|\nu| + i\nu\omega.$$

- Reduces to associated Laguerre equation

$$x\psi''_{\kappa,\nu} + (1+|\nu|-x)\psi'_{\kappa,\nu} + \kappa\psi_{\kappa,\nu} = 0.$$

Higher-order correlations?

- Does the KZ state avoid all divergences as $\gamma \to 0$?
- Choose $G = a_1^* a_1 a_2^* a_2$ in the Ehrenfest equation
- Similarly

$$\lim_{\gamma \to 0} \left\langle \frac{d}{dt} a_1^* a_1 a_2^* a_2 \right\rangle = \lim_{\gamma \to 0} 2 \left(\gamma_1 + \gamma_2 \right) \left\langle a_1^* a_1 a_2^* a_2 \right\rangle_c \neq 0?$$

- Divergence avoided for thermal equilibrium, but not KZ?