A Local Model for Strong Wave Turbulence

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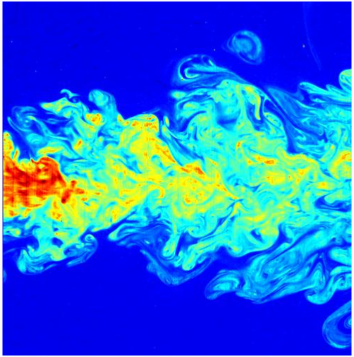
Energy cascade

- Energy flows from eddies at large scales (small k) to small scales (large k)
- Stochastic, but goal is to predict the mean *energy density distribution* $E(k)$
- The change of energy with time is determined by the *energy flux* P

$$\frac{d}{dt}\int_{k_i}^{k_f} E(k)\,dk = -(P_f - P_i)$$



Source: C. Fukushima, J. Westerweel

- If the flux P is constant, $E(k)$ reaches a stationary state

Kolmogorov spectrum

- We can find the stationary distribution $E(k)$ from dimensional analysis
  - $E(k)$ is energy density per wave number k
  - Flux P is energy density per time
  - Only other parameter is the fluid mass density ρ

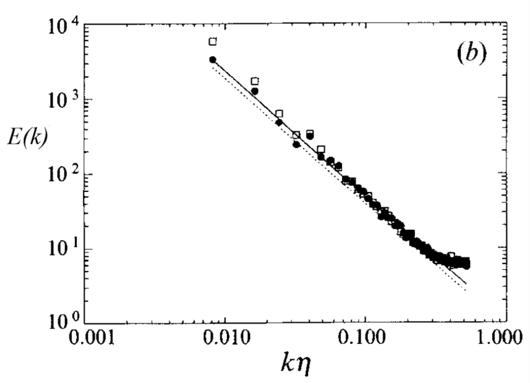
$$[k] = L^{-1}, \qquad [E(k)] = EL^{-2}, \qquad [P] = EL^{-3}T^{-1}, \qquad [\rho] = EL^{-5}T^{2}$$

- The only dimensionally consistent expression is

$$\boxed{E(k) \propto \rho^{1/3} P^{2/3} |k|^{-5/3}}$$

Experimental evidence of $5/3$ scaling

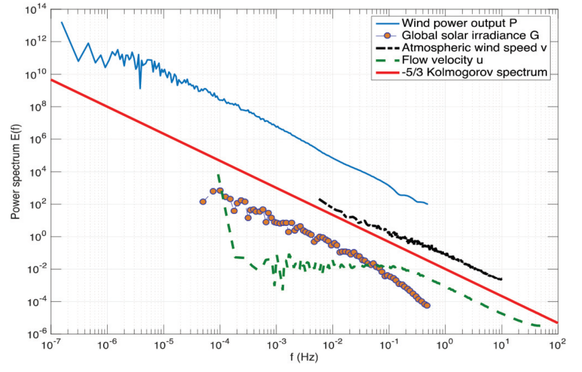
- Direct measurement of fluid velocities from fluorescent tracers



Liu, Meneveau, and Katz, *J. Fluid Mech.* **275** (1994).

Experimental evidence of $5/3$ scaling

- Relevance of the $5/3$ spectrum to renewable energy sources
- Frequency rather than wave number: Taylor's frozen-flow hypothesis



Calif, Schmitt, and Medina, 2016.

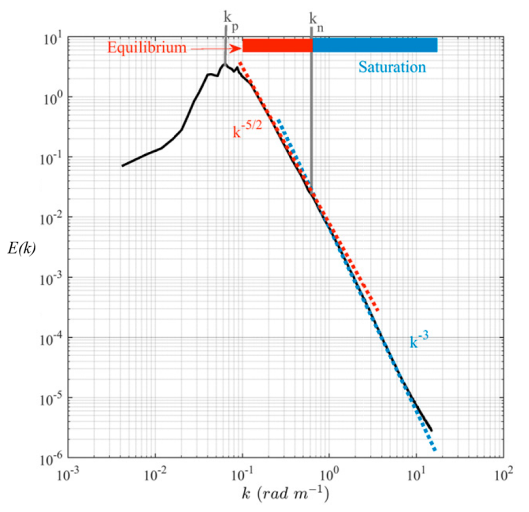
Fluid gravity wave turbulence

- The context of my work: turbulence of interacting waves
- *Gravity waves* involve an extra dimensionful parameter, g
- Can we still use dimensional analysis to find the spectrum?
- Almost: wave turbulence theory fixes the dependence on the flux P
- Kolmogorov–Zakharov (KZ) spectrum:

$$\boxed{E_{\mathrm{KZ}}(k) \propto \rho^{2/3} P^{1/3} g^{1/2} |k|^{-5/2}}$$

Experimental evidence of KZ scaling (5/2)

- Measured height of ocean surface with LiDAR
- Reconstructed the energy spectrum $E(k)$
- Large k: different scaling law?



Lenain and Melville, *J. Phys. Oceanogr.* **47** (2017).

Phillips spectrum?

- Postulated from dimensional analysis, with no dependence on the flux P
- Phillips spectrum:

$$\boxed{E_{\mathrm{Ph}}(k) \propto \rho g |k|^{-3}}$$

Source: recon.sccf.org

- Believed to involve highly nonlinear dynamics and wave breaking

Phillips spectrum from a local turbulence model

- We introduce a PDE for $E(k, t)$ that exhibits both KZ and Phillips spectra
- Schematically,

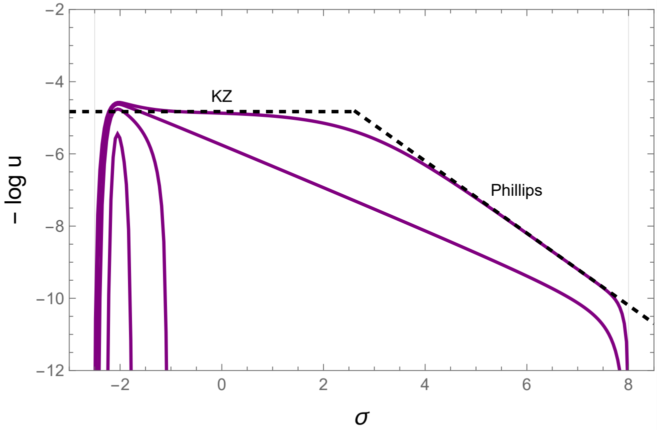
$$\frac{\partial E}{\partial t} = -\frac{\partial}{\partial k} P[k, E(k)]$$

- For clarity, work with the rescaled quantity $u^{-1} \propto k^{5/2} E(k)$
- The KZ spectrum is flat on a log-log plot

Approaching the stationary turbulent state

$$u^{-1} \propto k^{5/2} E(k)$$

$$\sigma = \frac{1}{2} \log k$$



daschubring.com/turbulenceAnimations.html

Levels of description of turbulence

0. **Experiment / observation**
1. **Basic physical equations**
   - e.g. Navier–Stokes, wave equations for gravity waves
   - Numerical simulation can provide important evidence
2. **Statistical description**
   - Wave kinetic equation
   - Methods similar to those in quantum field theory
3. **Reduced models**
   - e.g. differential approximation models
   - Can be derived from 2, or motivated directly from 0 or 1

Equations of motion for gravity waves

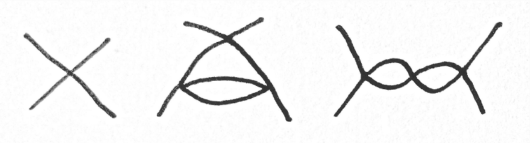
- Describe the fluid by the surface height η and velocity potential ψ
- The total energy is

$$H[\eta, \psi] = \int d^2x \left(\underbrace{\frac{\rho}{2} \psi \, G(\eta) \, \psi}_{\text{kinetic}} + \underbrace{\frac{\rho g}{2} \eta^2}_{\text{potential}} \right)$$

- In the limit of a flat surface, the equations of motion are linear
- Frequency $\omega = \sqrt{gk}$
- η dependence of kinetic term leads to nonlinearity. Interaction between waves

Feynman diagrams

- We can represent perturbation theory by diagrams as in particle physics
- As if waves interact in discrete 2-body collisions
- If nonlinearity is small, we only need diagrams with few vertices
- Textbook wave turbulence theory only involves the simplest diagram
- This leads to a kinetic theory of scattering waves



Local models for turbulence

- The kinetic theory involves an integro-differential equation
- Local approximation: waves of very different k don't interact
- Leads to a PDE for the energy spectrum

$$\partial_t E = -\partial_k \left(\underbrace{K - 2k\partial_k K}_{=P} \right)$$

- K encodes the underlying wave interactions ("level 2")
- In terms of $\omega = \sqrt{gk}$ and occupation number $n(k) \propto k^{-3/2} E(k)$,

$$K = \lambda^2 \; \omega^{24} n^4 \, \omega^2 \partial_\omega^2 n^{-1}$$

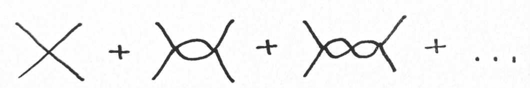
Dyachenko et al. *Physica D* **57** (1992)

Extending to strong turbulence

- Phillips turbulence is highly nonlinear. We cannot restrict to the simplest diagram
- Intuition: geometric series

$$\frac{x}{1-x} = x + x^2 + x^3 + \cdots$$

- We cannot deal with *all* diagrams, but we select an infinite class



- Each "bubble" contributes $\lambda_{bubble}\,\omega^{10}\partial_\omega n$
- K in the PDE is modified:

$$\lambda^2 \to \frac{\lambda^2}{\left|1 - \lambda_{bubble}\,\omega^{10}\partial_\omega n\right|^2}$$

Summarizing the PDE

$$\partial_t E = -\partial_k \left(K - 2k\partial_k K\right), \qquad K = \frac{\omega^{24} n^4 \, \omega^2 \partial_\omega^2 n^{-1}}{\left(1 + \omega^9 \omega \partial_\omega n\right)^2}$$

- K is described in terms of $n = k^{-3/2} E(k)$, $\omega = \sqrt{gk}$
- For simplicity, parameters $\lambda, \lambda_{bubble}$ are absorbed into scale of t, n
- How are the two forms of turbulent scaling seen at a glance?
  - KZ scaling $E(k) \sim k^{-5/2} \implies n(\omega) \sim \omega^{-8}$. Makes the numerator constant
  - Phillips scaling $E(k) \sim k^{-3} \implies n(\omega) \sim \omega^{-9}$. Can make the denominator vanish

Conclusion

- What we showed
    - A strongly local model can describe both KZ and Phillips turbulence
    - Higher-order wave interactions enter through a simple resummation
    - The resulting PDE reproduces the crossover dynamically
- Future directions in wave turbulence
    - How accurate is the local approximation? Consider lower levels of description
    - Capillary-wave turbulence and chiral turbulence (w/ S. Ganeshan and K. Kolluru)
    - Generative / diffusion models for turbulent wave fields

ODE for the stationary states

- If K is constant, flux $P = K - 2\partial_k K$ is also constant, and $E(k)$ is stationary
- The stationary equation $K = P$ is

$$\lambda^2 \omega^{26} n^4 \partial_\omega^2 n^{-1} = P \left|1 - \lambda c\, \omega^{10} \partial_\omega n\right|^2$$

- More transparent in variables

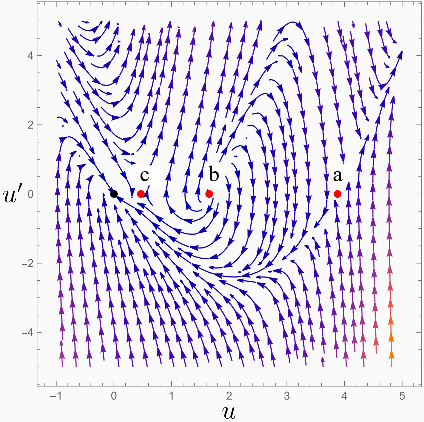
$$u \propto \frac{n_{\mathrm{KZ}}}{n}, \qquad \sigma = \log \frac{\omega}{\omega_0}$$

- Choosing the normalization of u and ω_0 appropriately,

$$\boxed{u'' + 15u' + 56u = \left[u^2 - e^\sigma \left(u' + 8u\right)\right]^2}$$

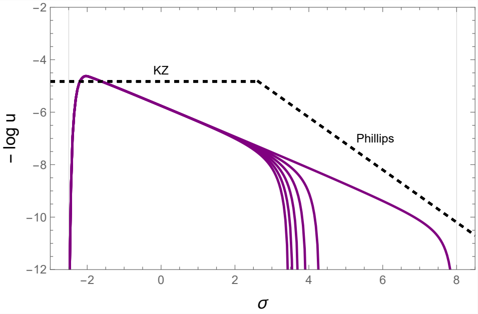
Phase portraits

$$u'' + 15u' + 56u = \left[u^2 - e^\sigma\left(u' + 8u\right)\right]^2$$



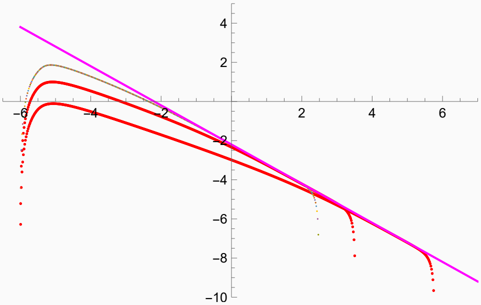
- Non-autonomous but examine the phase portrait at fixed σ
- For $e^\sigma \ll 1$ approximately autonomous. u_a is the KZ solution
- As σ increases, both fixed points $u_{a,b} \sim e^\sigma$. This is Phillips scaling in these coordinate
- There is a solution staying near a for all σ. This is the KZ $\to$ Phillips solution seen in the simulation.

Finite-time scaling



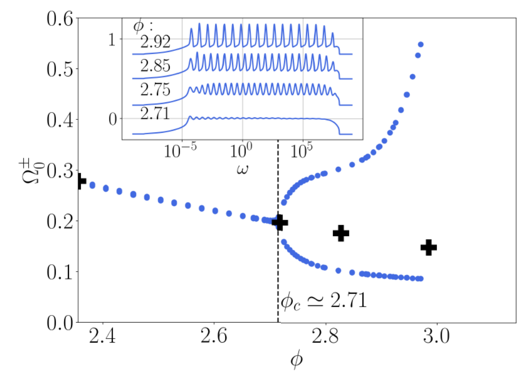
- The spectrum approaches a self-similar form as the front moves to large k
- Scaling ansatz $n(\omega, t) = (t_\star - t)^{bx} F(\eta)$ with $\eta = (t_\star - t)^b \omega$
- x is an anomalous scaling exponent related to the slope -0.583
- Substitution into the PDE gives a fourth-order ODE for $F(\eta)$
- The boundary conditions select the allowed value of x

Finite-time scaling: Phillips regime



- The anomalous front eventually may reach the Phillips regime
- The pink line has Phillips scaling $u(\omega) = \Omega_0\, \omega^9$
- Use the same self-similar ansatz but Phillips scaling fixes $x = 9$, $b = 1$
- Now the boundary conditions of the ODE determine Ω_0, rather than exponent x

Bifurcation diagram



- λ_{bubble} has residual phase $e^{i\phi}$. So far we took $\phi = \pi$
- Phillips coefficient Ω_0 depends on ϕ
- At $\phi_c \simeq 2.71$, the Phillips solution becomes oscillatory
- The extrema $\Omega_0^\pm$ form a bifurcation diagram

- We can still apply scaling ansatz $F(\eta, t)$ in the PDE and compare to ODE