Skyrmions and Hopfions in 3D Frustrated Magnets

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Outline

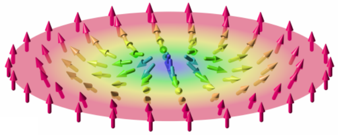
- Naya, D.S., Shifman, Wang (2021) arXiv: 2111.06385
- 3D topological defects in a lattice spin model
- Introduction
  - What is a 2D Skyrmion? (baby Skyrmion)
  - What is a 3D Skyrmion? (baryon)
  - A bit of geometry involving $S^3 \to S^2$
- Lattice spin models and baryons
- Extension of baby Skyrmions to 3D loops of string

Heisenberg model and 2D Skyrmions

- Classical ferromagnetic Heisenberg model

$$H = -J \sum_{<x,y>} S(x) \cdot S(y) \quad \sim \frac{J}{2} \int d^2x \, (\partial S(x))^2$$

- Target space is S^2. $\pi_2\left(S^2\right) = \mathbb{Z}$, so topological defects in 2D
- Derrick's theorem: Must be additional higher derivative terms
- Frustrated interactions rather than Dzyaloshinskii-Moriya
- "baby Skyrmions" or "magnetic Skyrmions"



[0] Image: Karin Everschor-Sitte and Matthias Sitte (Wikimedia Commons)

Principal chiral model and 3D Skyrmions

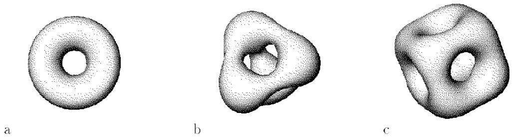
- Low energy effective theory of QCD in terms of $U \in SU(N_f = 2)$

$$\mathcal{L}_{PCM} = \frac{f_\pi^2}{4} \mathrm{Tr}\left(\partial_\mu U^{-1} \partial_\mu U\right).$$

- A column of U is a 2-component complex unit vector $z \in S^3$

$$U = \begin{pmatrix} \bar{z}^1 & z^0 \\ -\bar{z}^0 & z^1 \end{pmatrix}, \qquad \mathcal{L}_{PCM} = \frac{f_\pi^2}{2} \partial \bar{z} \cdot \partial z.$$

- Again topological defects since $\pi_3\left(S^3\right) = \mathbb{Z}$. Stable if Skyrme term
- $Q = 1$ 'hedgehog' solution. $Q > 1$ less symmetric rational map solutions



[0] Image: Houghton, Manton, Sutcliffe (1997)

Mapping S^3 to S^2

- Can relate the S^3 and S^2 sigma models more directly
- If we ignore the phase of z, this is CP^1. Bloch sphere
- More concretely, $S^i = -\bar{z}\sigma^i z$. Does not depend on phase
- If we use this change of variables,

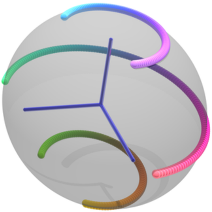
$$\frac{1}{4}(\partial S)^2 = \partial\bar{z}\cdot\partial z - |\bar{z}\cdot\partial z|^2$$

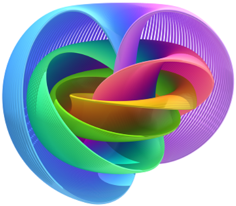
- Additional term subtracts out any change of z in phase direction
- Could imagine only partially subtracting out this change

Squashed sphere

- Hopf fibration visualization
- Generically $SU(2) \times U(1)$ global symmetry
- Can adjust length of fibers
- Zero length: $U(1)$ is gauged
- Special length: $SU(2) \times SU(2)$
- Intermediate length parametrized by κ

$$\partial \bar{z} \cdot \partial z - \kappa \left| \bar{z} \cdot \partial z \right|^2$$





[0] Image: Niles Johnson (Wikimedia Commons)

Physical relevance of squashed sphere

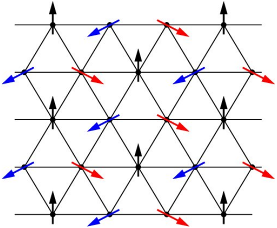
- In 2D, useful as a toy model for high-energy
  - Integrability: Cherednik '81, Kawaguchi, Yoshida '10
  - Prototype 'Yang-Baxter model,' (Review: Thompson '19)
  - Equivalent to the CP^N model coupled to extra fields
    - Stueckelburg field
    - Axion field (T-duality)
    - Massless fermion (bosonization)
- AdS_4/CFT_3 duality: Bykov '10, Basso, Rej '12, '13
- In 2D and 3D, as an EFT for condensed matter systems
  - Two-component LG models coupled to extra fields
  - **Frustrated spin systems**: Dombre, Read '88; Chubukov, Sachdev, Senthil '94; Azaria, Lecheminant, Mouhanna '95;

2D triangular antiferromagnet

- Coplanar ground state. Constant spins e_r on sublattices
- Slow variation in spins $S_r(x) = R(x)e_r, \quad R \in SO(3)$.
- Continuum effective theory: take $S_r(x)$ as defined at all x for all r.
- $S_r(x) \cdot S_s(x + a\delta_{r,s}) \sim S_r(x) \cdot \left(S_s(x) + \frac{a^2}{2!}\left(\delta_{r,s} \cdot \nabla\right)^2 S_s(x) + \dots\right)$

- In general upon summing δ spatial derivatives will not be rotationally symmetric. They are in this case
- Can be rewritten in terms of R field

$$-\partial S_r \cdot \partial S_s \quad \implies \text{Tr}\left[\left(R^{-1}\partial R\right)^2 e_s \otimes e_r\right]$$



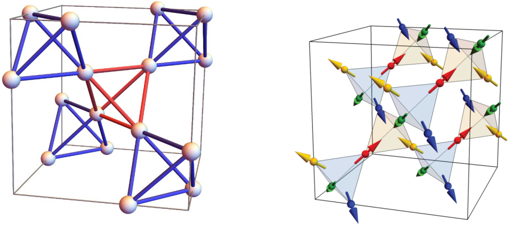
[0] Dombre, Read (1988), Figure: Chernyshev, Zhitomirsky (2009)

Right translation currents J_μ^k

- $\mathcal{H} \propto \mathrm{Tr}\left[\left(R^{-1}\partial R\right)^2 P\right], \quad P \equiv \sum_{s,r} e_s \otimes e_r$
- Lie algebra valued current $\left(R^{-1}\partial_\mu R\right)_{ij} = 2\epsilon_{ijk} J_\mu^k$
- Gives rate of change on Lie group in a given direction μ
- For typical lattices $P = \mathrm{diag}(a, a, b), \quad a \neq b$ implies squashing!
- $\mathcal{H} \propto \left(J_\mu^1\right)^2 + \left(J_\mu^2\right)^2 + (1-\kappa)\left(J_\mu^3\right)^2$
- At $\kappa = 1$ motion in J^3 direction not penalized
- Gauge transformations: J^1, J^2 rotate by 2ϕ, $J_\mu^3 \to J_\mu^3 + \partial_\mu \phi$
- 3D Skyrmion (baryon) charge from Jacobian:

$$Q \propto \int d^3x\, \epsilon^{\lambda\mu\nu} J_\lambda^1 J_\mu^2 J_\nu^3$$

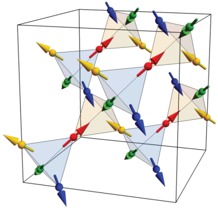
Pyrochlore lattice



- Rather than triangles, tetrahedra
- Macroscopic degeneracy of classical ground state
- To break this:
  - Higher order neighbor interactions, especially third
  - Include quartic terms from the microscopic model $\implies$ "all-in-all-out"

[0] Batista, Shifman, Wang, Zhang (2018)

Toy lattice model



- Triangular or tetrahedral cells are abstracted to a single point
- Rectangular lattice, three spins e_r per site
- Angle between spins fixed by a constraint
- More colinear = more squashed
- Spin e_r only interacts with spin with same r

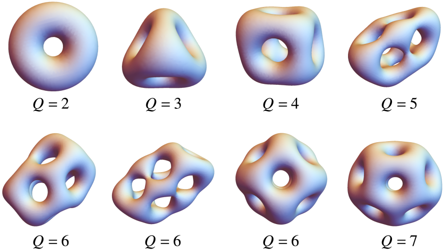
Frustration and stability of defects

- The continuum limit is the squashed sphere model
- Derrick's theorem: We need to keep higher order derivative terms

$$S_r(x + a\delta_r) = S_r(x) + \frac{a^2}{2!}\left(\delta_r \cdot \nabla\right)^2 S_r(x) + \frac{a^4}{4!}\left(\delta_r \cdot \nabla\right)^4 S_r(x) \dots$$

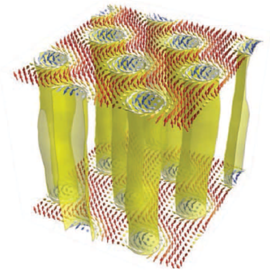
- But still a sign problem from integration by parts
- Consider ferromagnetic interaction at a and antiferromagnetic at $2a$
- Inversion-symmetric frustrated magnets: Lin, Hayami '16; Sutcliffe '17
- Lifschitz transition: Coefficient of 2nd order terms goes to zero
- Side note: Squashed 4th order terms involve gauge potential J^3

Results of lattice simulation



- Very similar to Skyrme model, BPS monopoles
- Rational map ansatz: $U(x) = \cos f(r) + i \sin f(r) \hat{n}(\theta, \phi) \cdot \vec{\sigma}$
- Energy and top charge density of ansatz very close to true solution
- Solutions robust under change of initial conditions
- Switching gears...

Magnetic Skyrmion strings



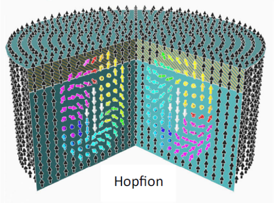
- In 3D, baby Skyrmions are extended
- 1d core is called 'position curve'
- 2-form to measure baby Skyrmion charge

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{ijk} S^i \partial_\mu S^j \partial_\nu S^k$$

- Can show $dF = 0$ since $S \cdot \partial S = 0$
- Sum of baby Skyrmion charge on a closed surface is zero
- Loops aren't topologically stable?

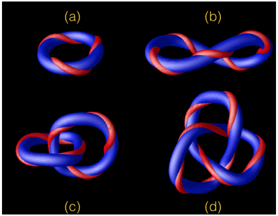
[0] Milde et al (2013); Seki et al (2021)

Hopfions



- Just twist the string or tie it in a knot!
- On S^3, globally defined gauge field $F = dA$
- The Hopf charge is defined as

$$Q = -\frac{1}{8\pi^2} \int d^3x \, \epsilon^{\lambda\mu\nu} A_\lambda F_{\mu\nu}$$



- Frustrated magnet has Hopfion solutions!
- (Another story: Faddeev-Niemi model and the squashed Skyrme model)

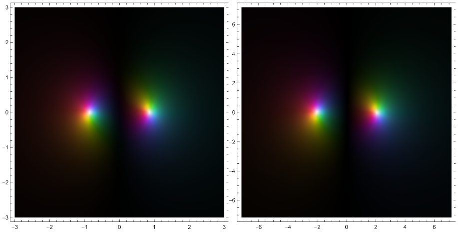
[0] Sutcliffe (2017); Kent et al (2021)

Hopfion charge and baryon charge

- What is the relation between the 3D Skyrmions and Hopfions?
- Gauge invariant dJ^3 should be expressible in terms of S^i
- In fact $dJ^3 = F$! So $A = J^3$
- Another gauge invariant quantity is $J^1 \wedge J^2$
- Flat connection: $F = dJ^3 = 2J^1 \wedge J^2$
- So both charges are the same thing! $A \wedge F \propto J^1 \wedge J^2 \wedge J^3$
- Any field configuration to S^3 can be projected to S^2 with same Q
- Likewise we may (non-uniquely) extend any field configuration to S^2
- The only difference is energy associated to J^3 field

Hedgehog Skyrmions and position curve loops

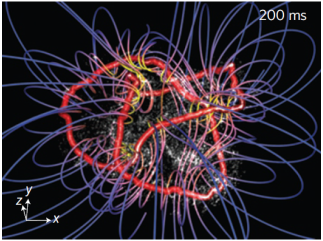
- How can a spherically symmetric ‘nucleon’ be a loop of string?
- The energy associated to J^1, J^2 concentrated near loop
- J^3 flows through the middle like a dipole field
- The energy and topological charge densities take both into account
- As we ‘squash’ the system, only very subtle change



Thin 'baryon strings'

- Dipole analogy: The baby Skyrmion charge is like 'current'
- Since $F = dJ^3$, the circulation of J^3 is proportional to 'current'
- From Euler-Lagrange, $\nabla \cdot J^3 = 0$ outside core
- Can use Biot-Savart law to find J^3 outside core (fluid analogy)

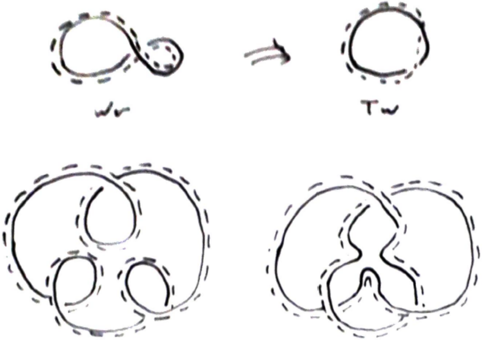
- Unwind hedgehog to form straight string
- Global vs local strings
- Energy of large loops agrees with unstable solutions in Battye, Sutcliffe (1999)
- Strings lower energy by 'crumpling,' $E \sim Q^{3/4}$



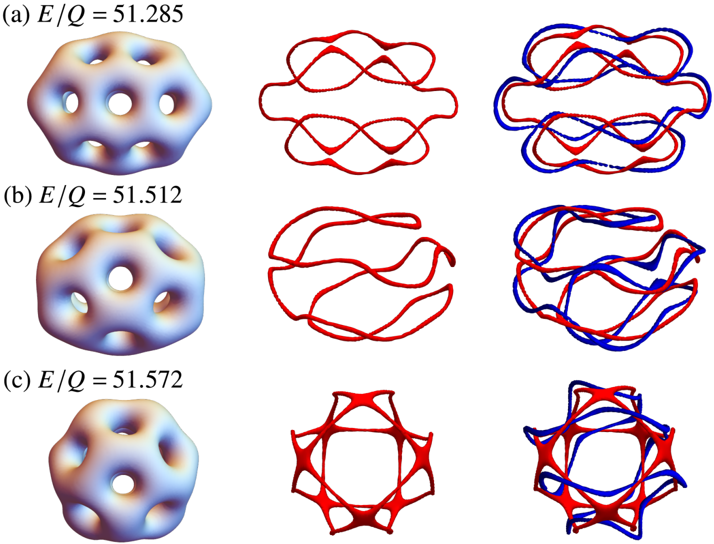
[0] Image: Kleckner, Irvine (2013)

Nuclei as vortex knots?

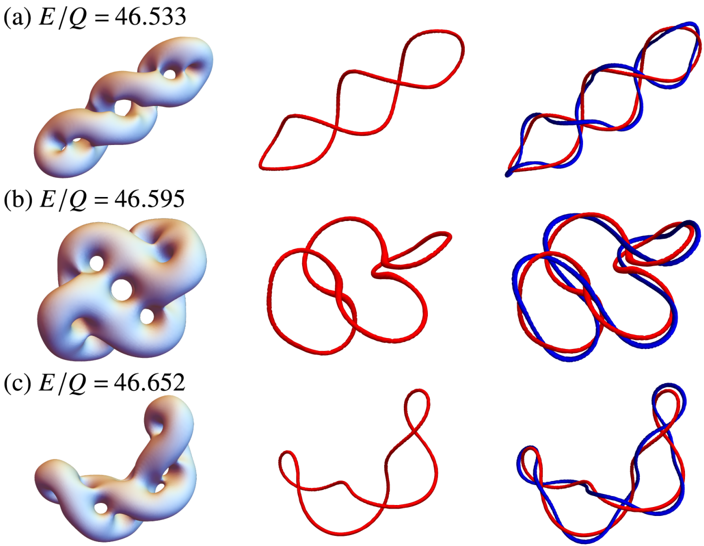
- Lord Kelvin's old idea
- How can isolated loops form a knot?
- Consider 'ribbon,' $Q = Tw + Wr$



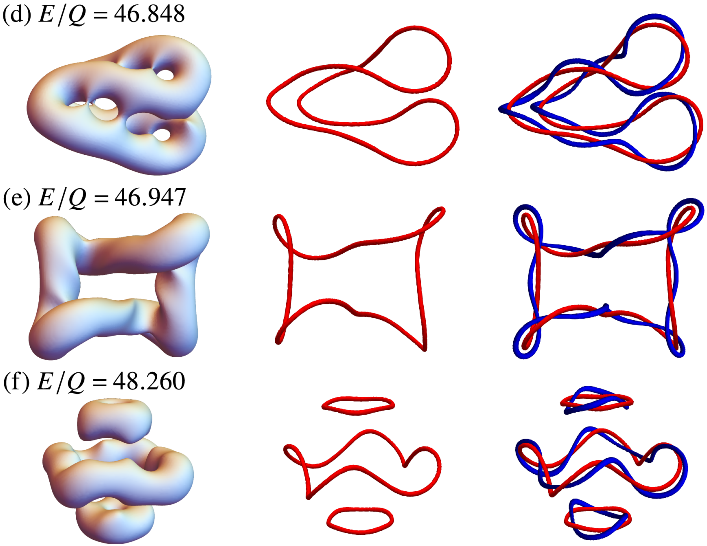
Charge 10 at $\kappa = 0$



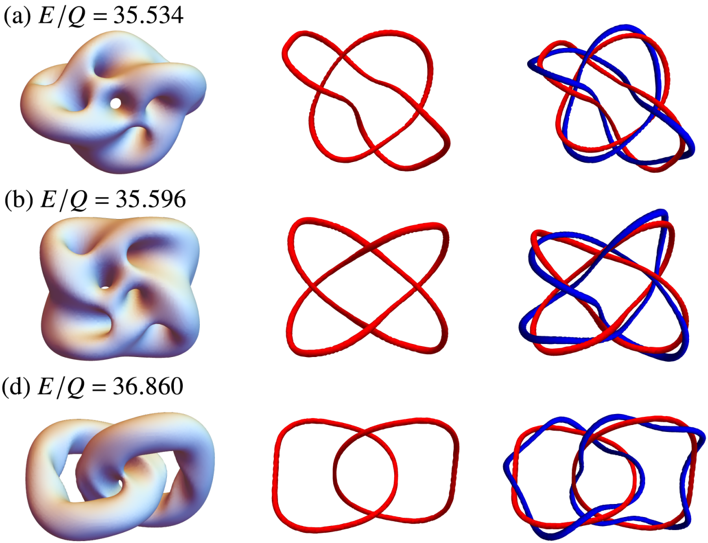
Charge 10 at $\kappa = 5/7$



Charge 10 at $\kappa = 5/7$

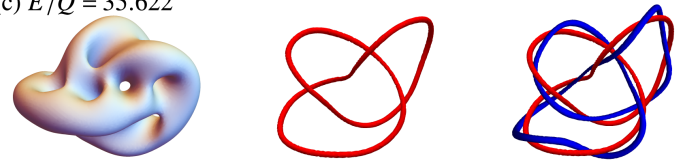


Charge 10 at $\kappa \approx 1$



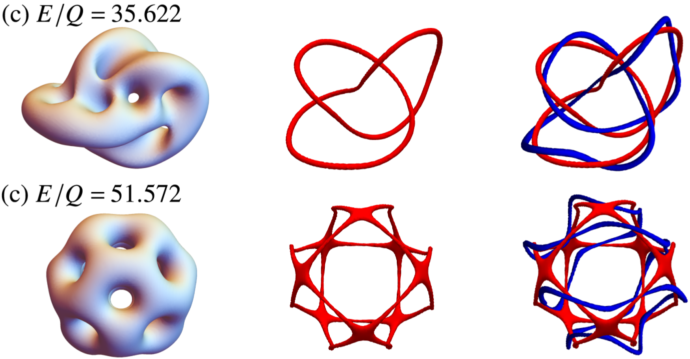
Conclusion

(c) $E/Q = 35.622$



- Future directions
  - Exploring metastable states as κ is varied in more detail
  - 3D Skyrmions in real materials. Pyrochlore lattice?
  - Lifschitz transition and Skyrmion crystals
  - Any insight into the Skyrme model and QCD?

Thank you!



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