Sigma models on fiber bundles

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What is a nonlinear sigma model?

- Simplest example: O(3) model

$$\mathcal{L} = \frac{1}{2\lambda^2}\partial^\mu S_a \partial_\mu S_a$$

- $S_a(x)$ is a constrained real field $S_1^2 + S_2^2 + S_3^2 = 1$
- Fields map space-time to the *target space* on S^2
- (Haldane) In 1+1D, O(3) model $\sim$ Heisenberg antiferromagnet
- For our purposes, it is a good toy model
- Rather than constrained field, can choose coordinates ϕ^a

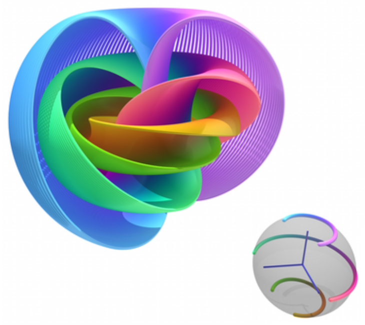
$$\mathcal{L} = \frac{1}{2\lambda^2} g_{ab}(\phi)\partial^\mu \phi^a \partial_\mu \phi^b$$

Two nice families of spaces

- S^2 can be extended in two distinct ways
- Target spaces on S^{N-1} referred to as $O(N)$ models
  - Maximally symmetric
  - Integrable
  - For $S^3 \sim SU(2)$ can define WZW term
- S^2 is also the Bloch sphere
- Take N normalized complex n_a. $n_a^2 = 1$. Mod out phase
- Target space on CP^{N-1}
  - Kahler, extended SUSY
  - Confinement
  - Can define theta term

Hopf fibration

- The complex $n_a^2 = 1$ also defines S^{2N-1}
- Modding out the phase gives fiber bundle $S^{2N-1} \to CP^{N-1}$
- Fibrated CP^{N-1} model:
  - Vary the size of the phase direction in metric
  - Interpolates smoothly between 2 toy models
  - Global symmetry $SU(N) \times U(1)$



Sigma models on fiber bundles to CP^{N-1}

- Condensed matter applications for fiber bundles to CP^1
  - Antiferromagnets on a 2D triangular lattice. Dombre, Read (1989)
  - Reconsidered, also on 3D pyrochlore lattice. Batista et al (2018)
- These systems have $SU(2) \times SU(1)$ symmetry
- We will stick to 1+1D in this talk, general N
- Outline:
  - Introduce Lagrangian
  - Renormalization to 1 loop and correlation functions
  - Renormalization from a more geometrical perspective
  - Topological terms

Introducing Lagrangian

- S^2 has 3-component real S,

$$\mathcal{L}_{S^2} = \frac{1}{2\lambda^2}\partial S \cdot \partial S, \quad S \cdot S = 1$$

- Choose 2-component complex n,

$$n^\dagger \sigma_a n \equiv S_a, \quad n^\dagger \cdot n = 1$$

- Lagrangian becomes gauge invariant $n \to n e^{i\phi(x)}$

$$\mathcal{L}_{S^2} = \frac{1}{2\lambda^2}\left(\partial n^\dagger \cdot \partial n - |n^\dagger \partial n|^2\right)$$

- Simply introduce parameter κ,

$$\mathcal{L}_{\kappa} = \frac{1}{2\lambda^2}\left(\partial n^\dagger \cdot \partial n - \kappa |n^\dagger \partial n|^2\right)$$

- For $\kappa = 0$, this is S^3, for $\kappa = 1$ this is CP^1

More on CP Lagrangian

- This introduction of κ might seem ad hoc
- Can write in a way that makes gauge symmetry more obvious

$$\mathcal{L} = \frac{1}{2\lambda^2}(\partial_\mu - iA_\mu)n^*(\partial^\mu + iA^\mu)n$$

- A_μ is auxiliary gauge field, fixed by eq of motion

$$A_\mu = \frac{i}{2}(n^*\partial_\mu n - \partial_\mu n^*\, n) \qquad A_\mu \to A_\mu - \partial_\mu \theta$$

- Looks like charged scalars n interacting with electric field
- Auxiliary but due to loops of n, A becomes effectively dynamical
- In 1 spatial dimension, Gauss's law implies electric field constant
- Confines n particles into mesons
- Coupling to Stuckelberg field, massless Dirac Fermion $\to$ mass term $\frac{1-\kappa}{\kappa}A_\mu^2$
- Mass term screens confinement at large distances

Metric from Lagrangian

$$\mathcal{L}_\kappa = \frac{1}{2\lambda^2}\left(\partial n^\dagger \cdot \partial n - \kappa |n^\dagger \partial n|^2\right)$$

- This was introduced for $S^2 \sim CP^1$, but extends to any N.
- Recall in general the Lagrangian involves metric on target space
- Extra term just cancels the metric component when ∂n points along phase
- So the parameter κ determines the distance along the fibers
- The parameter λ determines overall size
- Of course there is implicitly also a cutoff scale
- As we adjust this scale κ, λ must change to keep the same physics
- How does the shape of target space change with renormalization?
- Renormalization $\sim$ modified scaling symmetry. Callan-Symanzik

Wilson renormalization

- General approach: Original fields in action, ϕ_0. Cutoff at μ_0

$$\mathcal{L} = \frac{1}{2} g_{ab}(\phi_0) \partial \phi_0^a \partial \phi_0^b$$

- Now consider $\phi_0 = \phi + q$
    - ϕ is coarse grained up to lower cutoff μ
    - q is fluctuations near μ, integrate out of action
- The relevant terms involving q:

$$\mathcal{L}^{(2)} = \frac{1}{2} \left(g_{ab}(\phi)\, \partial^\mu q^a \partial_\mu q^b \; + 2\partial_c g_{ab}(\phi) \partial^\mu \phi^a \; q^c \partial_\mu q^b + \frac{1}{2!} \partial_d \partial_c g_{ab}(\phi) \partial^\mu \phi^a \partial^\mu \phi^b \; q^c q^d \right)$$

- We get the correct structure $\partial_\mu \phi^a \partial^\mu \phi^b$, loops produce $\log \frac{\mu}{\mu_0}$
- Right idea. Correct renormalization for S^2... but not S^3 or higher
- Curiously, only for Kahler target spaces. What went wrong?

Polyakov renormalization

- Need to keep the correct symmetries when we separate out fluctuations q
- Can keep manifest $SU(N) \times U(1)$ symmetry a la Polyakov
- Unit vector complex n fields. Real σ, N-1 complex q fluctuations

$$n_0 = e^{i\sigma}\sqrt{1-|q|^2}n + q^a e_a$$

- e_a is an arbitrary basis, orthonormal to n
- Can choose differently at each point $\rightarrow SU(N-1)$ gauge symmetry
- Similar calculation to last slide. This time correct RG equations
- We can also calculate field renormalization

$$\langle n_0 \ldots \rangle = \left[1 - \frac{\lambda^2}{4\pi}\left(\frac{1}{1-\kappa} + 2(N-1)\right)\log\frac{\mu_0}{\mu}\right]\langle n \ldots \rangle$$

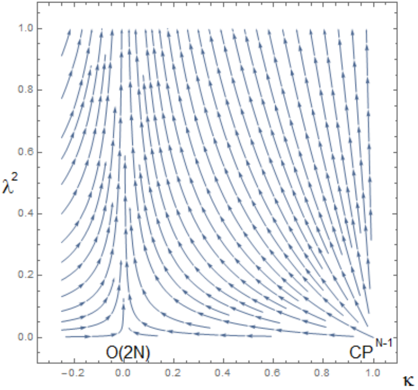
RG equations

$$\mu\frac{\partial}{\partial\mu}\lambda^2 = \frac{\lambda^4}{2\pi}2(N-1+\kappa), \qquad \mu\frac{\partial}{\partial\mu}\kappa = -\frac{\lambda^2}{2\pi}2N\kappa(1-\kappa)$$

- Not first to find these (Azaria et al, 1995)

- Recall λ^2 related to overall size, κ related to size of fibers
- Arrows are pointing to IR
- Asymptotically free in UV
- IR scale Λ at which perturbation theory breaks down
- Trajectories labeled by RG invariant

$$K = \frac{\kappa^{1-\frac{1}{N}}}{1-\kappa}\lambda^2$$



2-point correlation function

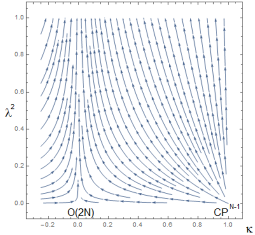
- What is $K = \frac{\kappa^{1-\frac{1}{N}}}{1-\kappa}\lambda^2$?
- We found anomalous dimension γ of n. As $\lambda^2 \to 0$, $\kappa \to 1$,

$$\gamma = \frac{\lambda^2}{4\pi}\left(\frac{1}{1-\kappa} + 2(N-1)\right) \to \frac{K}{4\pi}$$

- So power law in UV

$$\langle n^\dagger(p) \cdot n(-p)\rangle \sim \frac{1}{p^2}\left(\frac{p^2}{\Lambda^2}\right)^{\frac{K}{4\pi}}$$

- For low K can crossover to $O(2N)$
- K could be sign of these $SU(N) \times U(1)$ systems



Covariant renormalization

- Polyakov style renormalization got us anomalous dimension
- But renormalization of general 2D sigma models is solved problem!
- Polyakov-style method kept global $SU(N) \times U(1)$
- Instead we could keep covariance on target space. Ensure result of renormalization does not depend on coordinates (Honerkamp,Ecker 1971)
- Fluctuating fields q don't transform covariantly
- Instead expand in terms of tangent space vectors at the point ϕ
- Tangent vectors do map to q through exponential map
- Result is well known

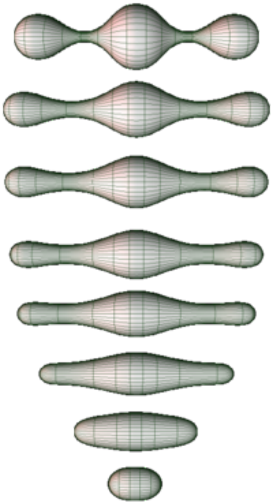
$$\frac{\partial}{\partial \log \mu} g_{ab} = \frac{1}{2\pi} R_{ab}$$

Ricci flow

- Same Ricci flow as in Perelman and Poincare conjecture
- More homogenous and smaller in IR, generically strong coupling
- What if fixed point? $R_{ab} = 0$
- Relevant to string theory
- At two loops, 'quantum correction'

$$\beta_{ab} = \frac{1}{2\pi} R_{ab} + \frac{1}{8\pi^2} R_a{}^{cde} R_{bcde}$$

- Topological term $\implies$ 'matter'



How to find Ricci tensor?

- Simply choose coordinates, calculate connection coefficients
- But this is tedious, can be difficult to generalize N and invert metric
- Use symmetry of the space. Go from $n \in S^{2N-1}$ to $U \in SU(N)$

$$Un_0 = n$$

- For $N = 3$, substituting into Lagrangian,

$$\mathcal{L} = \frac{1}{\lambda^2} \sum_{a=4}^{7} (J_\mu^a)^2 + \frac{4}{3} \frac{1-\kappa}{\lambda^2} (J_\mu^8)^2$$

- J_μ^a component of $\partial_\mu n$ in left-invariant vector field associated to τ^a

$$J_\mu^a = -\frac{i}{2} Tr(U^\dagger \partial U \tau^a)$$

- The metric is very simple in this basis

Fiber bundles

- Notice there is no dependence on J^1, J^2, J^3
- U is not uniquely specified by n
- $Un_0 = n$ fiber bundle projection map $SU(N) \to S^{2N-1}$
- The fibers are $\sim SU(N-1)$, same $SU(N-1)$ gauge symmetry as Polyakov
- Our choice of gauge is a section $\sigma(n) = U$
- This can be thought of as locally embedding S^{2N-1} as submanifold in $SU(N)$
- S^{2N-1} inherits geometry from $SU(N)$ via pullback
- Gauge invariance: No matter which embedding, same geometry on S^{2N-1}!
- If we find curvature of $SU(N)$ this implies curvature on S^{2N-1} as a submanifold. Gauss equation

Geometry from structure coefficients

- Metric diagonal in left-invariant basis $g(\tau_a, \tau_b) = \frac{1}{\lambda^2} C_a \delta_{ab}$
- Lie bracket of τ_a vector fields prop to matrix commutator

$$[\tau_a, \tau_b]_L \equiv \nabla_{\tau_a} \tau_b - \nabla_{\tau_b} \tau_a = -2 \sum_c f_{abc} \tau_c.$$

- This and metric compatibility determine covariant derivative

$$g(\nabla_{\tau_a} \tau_b, \tau_c) = -g(\tau_b, \nabla_{\tau_a} \tau_c) = -g(\tau_b, \nabla_{\tau_c} \tau_a) - g(\tau_b, [\tau_c, \tau_a]) = \dots$$

- Ultimately find Riemann and Ricci tensor (χ_a indicator function)

$$R_{aa} = \sum_{b,c} f_{abc}^2 \left(1 + \frac{C_b - C_a}{C_c} \chi_c + 3 \frac{C_a - C_c}{C_b} \chi_b - \frac{C_b - C_a}{C_c} \frac{C_a - C_c}{C_b} \chi_b \chi_c \right)$$

- Not surprisingly method not new, e.g. (Camporesi, 1990)
- But used it to find RG equations of fiber bundle to Grassmannian to 2 loops (D.S., Shifman, 2019)

Topological terms

- S^{2N-1} and CP^{N-1} differ in their topology
- Finite action field configuration $\rightarrow$ space-time $\cong S^2$
- $\pi_2\left(CP^{N-1}\right) = \mathbb{Z}$. Embedded space-time $\hat{\phi}$ can have winding number Q
- Adding theta term $i\theta Q$ to the action changes physics

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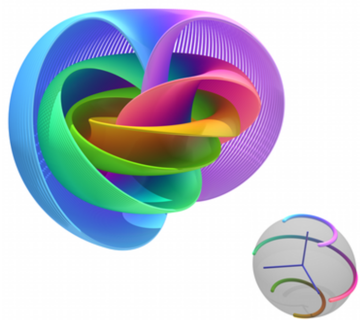
- (Haldane 1983) Half-integer Heisenberg antiferromagnet $\sim CP^1_{\theta=\pi}$
- (Affleck, Haldane 1987) $CP^1_{\theta=\pi} \rightarrow SU(2)_1\, WZW$
- This looks like our RG flow $CP^1 \rightarrow S^3$. Relation?

Theta term possible?

- Can we define a theta term on the fiber bundle to CP^1?
- Of course $\pi_2(S^3) = 0$
- But using fiber bundle $S^3 \to CP^1$ it is at least sensible
- Problem is surface $\hat{\phi}$ attaches to great circle in S^3
- Non-closed surface $\to$ divergent action
- Also seen from $\int dx^2 A^2$, consistent with massless fermions



Wess-Zumino-Witten term

- But if closed surface $\hat{\phi}$, we can integrate over interior
- Ambiguity, $\int dV$ or $\int dV \; - V$?
- Normalize volume form to $2\pi k$, integer *level* k

$$F_{abc} = \frac{2\pi k}{V}\sqrt{g}\epsilon_{abc}$$

- No ambiguity in WZW term in action $i\int F$,
- Dependence on size of fibers κ cancels in $\sqrt{g}/V$
- Since $dF = 0$, locally $F = dB$, 2-form gauge field
- Stokes' theorem $\int F = \int_{\hat{\phi}} B$

$$i\int dx^2 B_{ab}\epsilon^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b$$

- Can be treated by same RG methods as metric term

WZW fixed point

- Including volume form $F = dB$, Ricci flow changes

$$\frac{\partial}{\partial \log \mu} g_{ab} = \frac{1}{2\pi}\left(R_{ab} - F_a{}^{cd} F_{bcd}\right), \qquad \frac{\partial}{\partial \log \mu} B_{ab} = \frac{1}{2\pi} \nabla^c F_{cab} = 0$$

- 2nd equation like Maxwell's equation. Identitically zero
- 1st equation has fixed point, like gravity coupled to 2-form gauge field
- Since for S^3, $R_{ab} \propto g_{ab}$ and $F_a{}^{cd} F_{bcd} \propto g_{ab}$, we can solve

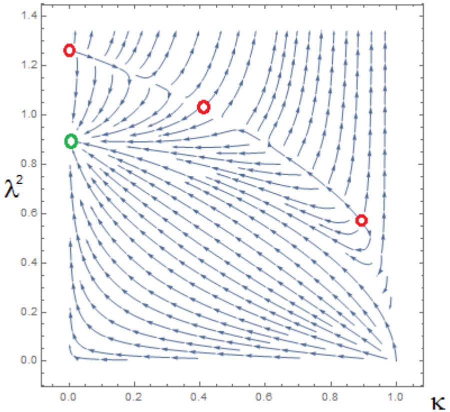
$$\lambda_k^2 = \frac{2\pi}{k}$$

- At this CFT can use a large class of symmetries related to our $J = -iU^\dagger \partial U$
- But for general κ, not Einstein manifold $\rightarrow$ flows

RG flow with WZW term

- Calculated to 2 loops
- Loop expansion is $1/k$ expansion. Only trust for high level k
- Dependence on λ^2/λ_k^2 and $(1-\kappa)/\lambda^2$ is more subtle

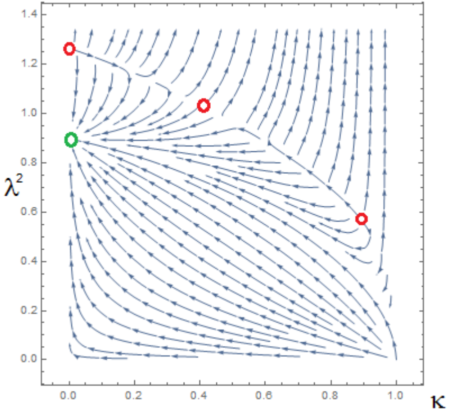
- Below first separatrix, seen at 1 loop
- At $\lambda^2 \ll \lambda_k^2$, previous RG diagram
- Scaling in UV and IR
- Can use CFT at λ_k^2 to test RG flow
  - Beta functions agree with exact scaling dimensions
  - $\lambda^2 = \lambda_k^2(1-\kappa)$ exact trajectory?



Extra fixed points?

- At second loop, see second separatrix
- 'Asymptotic safety,' non-trivial UV fixed points?

- For $k > 8$, interior fixed points disappear
- At $\kappa = 0$, 3-loop results exist (Ketov et al, 1989)
- Unstable fixed point still there, and new stable fixed point appears
- Ambiguities in 3-loop calculation
- Still unclear whether artifact of perturbation theory



Wrapping up

- Future directions
  - Compare to conformal perturbation theory about stable WZW CFT?
  - Wakimoto representation of WZW CFT
  - Exact in k, but perturbative in λ^2, κ
  - Could fix ambiguities in 3-loop calculation
- What's the point?
  - Connection to the Haldane conjecture ultimately misleading
  - But still very symmetric UV completion(s?) of WZW models, could turn up as effective theories
  - A road towards more realistic condensed matter systems?
    - Could explore 1+1d WZW $\sim$ 2+1d Chern Simons duality
    - High temperature correlators of 2+1d antiferromagnets?