Renormalons in 2D asymptotically free theories

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D.S, Sheu, Shifman (arXiv:2107.11017), D.S (arXiv:2207.10549)

Outline

- Introduce general ideas about resurgence
  - Borel transform
  - Renormalons
  - Transseries
- Introduce 2D model as a toy model for gauge theory
  - Variants of O(N) nonlinear sigma model
  - OPE of spinon correlation function is a transseries
- Find OPE and verify cancelation of ambiguities
  - OPE via background field method
  - OPE via expansion of large N result

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Divergent perturbation series

- Perturbation series are typically asymptotic series
- Anharmonic oscillator in QM

$$V = \frac{1}{2}\omega^2 q^2 + \frac{1}{3}\omega^3 \lambda q^4$$

- Can calculate ground state energy

$$E_0 = \frac{1}{2}\omega + \omega \sum_{n=0} A_n \lambda^{n+1}$$

- Bender, Wu (1969): alternating factorial

$$A_n \sim \sqrt{\frac{6}{\pi^3}}(-1)^n \Gamma\left(n + \frac{3}{2}\right)$$

- Dyson argument

Borel transform

- How can we treat asymptotic series?
- Define new series in complex t that is better behaved

$$I(\lambda) = \sum_{n=0} A_n \lambda^{n+1} \qquad \rightarrow_{\mathcal{B}} \qquad f(t) = \sum_{n=0} \frac{A_n}{n!} t^n$$

- Can recover original series

$$I(\lambda) = \int_0^\infty dt f(t) e^{-\frac{t}{\lambda}}$$

Borel transform: Example

$$I(\lambda) = \sum_{n=0} A_n \lambda^{n+1} \qquad \to_{\mathcal{B}} \qquad f(t) = \sum_{n=0} \frac{A_n}{n!} t^n$$

$$I(\lambda) = \int_0^\infty dt f(t) e^{-\frac{t}{\lambda}}$$

- Example: alternating factorial

$$\sum_{n=0}^{\infty} n!(-1)^n \lambda^{n+1} \qquad \to_{\mathcal{B}} \qquad \sum_{n=0} (-t)^n = \frac{1}{1+t}$$

- $I(\lambda)$ is well defined for any λ.

$$I(\lambda) = \int_0^\infty dt \frac{e^{-\frac{t}{\lambda}}}{1+t} = -e^{\frac{1}{\lambda}} \mathsf{Ei}\left(-\frac{1}{\lambda}\right)$$

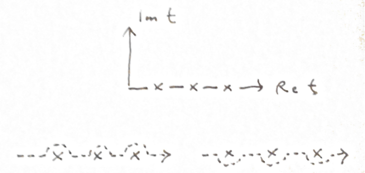
- Non-alternating factorial?

Renormalons

- Non-alternating factorial: Pole on the integration path

$$I(\lambda) =_? \int_0^\infty dt \frac{e^{-\frac{t}{\lambda}}}{1-t}$$

- Regulate by shifting contour slightly above and below



- 'Renormalon ambiguity'
- Total ambiguity $2\pi i \, e^{-1/\lambda}$
- Does this hint at non-perturbative physics?

Resurgence in the double-well

- Double-well potential in QM

$$V = \frac{\omega^2}{2} q^2 \left(1 - \sqrt{\lambda\omega} q\right)^2$$

- Brezin, Parisi, Zinn-Justin (1977)

$$A_n \sim -\frac{3}{\pi} n! 3^n$$

- Need to account for instantons in the path integral: tunneling
- Perturbation series including multi-instanton sectors: ‘transseries’
- Bogomolny (1980): Multi-instantons have ambiguities

Resurgence paradigm in QFT

- There is renormalon in QCD. Is it also cured by instantons?
- Dunne, Unsal (2012): Compactify with twisted boundary conditions
    - Instantons fractionalize
    - No phase transition with compactification scale
    - Reduces to ordinary QM
- We're taking different approach
    - No compactification
    - Correlation functions rather than free energy
    - Non-perturbative data is coming from expectation values of operators

$O(N)$ model

- Classical N-component unit vector spin n^i at each point

$$S = \frac{1}{2\lambda}\int d^2x \left(\partial n^i(x)\right)^2, \qquad \left(n^i\right)^2 = 1$$

- "N-vector model. Quantum rotor model. Sigma model on S^{N-1}"
- Perturbative parameter λ is like temperature.
- Choose unconstrained coordinates φ^a

$$n^i = \left(\sqrt{\lambda}\varphi^a, \sigma\right), \qquad \sigma \equiv \sqrt{1 - \lambda\left(\varphi^a\right)^2}$$

$$S = \frac{1}{2}\int d^2x \left(\partial\varphi^a\right)^2 - \lambda^{-1}\sigma\partial^2\sigma$$

- Looks like massless theory

Non-perturbative aspects of $O(N)$ model

- Finite correlation length m^{-1} depending on UV cutoff M

$$m = Me^{-\frac{2\pi}{\lambda}}$$

- Mermin-Wagner theorem
- Introduce Lagrange multiplier field $\alpha(x)$

$$\mathcal{L} = \frac{1}{2\lambda}\left[(\partial n)^2 + \alpha\left(n^2 - 1\right)\right]$$

- VEV of α (or $n \cdot \partial^2 n$) leads to mass term

$$\langle\alpha\rangle = m^2 + \mathcal{O}(1/N)$$

- Non-perturbative data will come from VEVs

Operator product expansion

- Lowest order correlation function $\langle n(p) \cdot n(-p) \rangle^{(0)}$

$$\frac{1}{p^2 + m^2} = \frac{1}{p^2} \sum_{k=0} \left(-\frac{m^2}{p^2} \right)^k = \sum_{k=0} (-1)^k p^{-2(k+1)} \langle \alpha^k \rangle$$

- Schematically

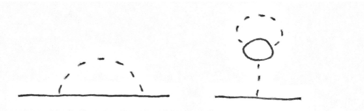
$$\langle n(p) \cdot n(-p) \rangle = \sum_k C_k(p, \lambda) \langle O_k \rangle$$

- "Coefficient functions" C_k involve a power series in λ at higher order
- Non-perturbative dependence on λ is all in operator VEVs

$$\langle O_k \rangle \propto m^{2k} \sim e^{-\frac{4\pi k}{\lambda}}$$

Renormalons in the $O(N)$ model

- To see renormalon ambiguities we must consider $\mathcal{O}(1/N)$ corrections



- David (1984): There are also ambiguities in the operator VEVs
- This was controversial in the mid 1980s. Two approaches:
  - Novikov, Shifman, Vainshtein, Zakharov (ITEP group)
    - Get OPE from a background field method
    - No ambiguities, but arbitrary factorization scale μ
  - David (1984,1986); Beneke, Braun, Kivel (1998)
    - Get OPE by expanding large N results
    - Ambiguities. No factorization scale

ITEP group approach

- 4D gauge theories and the 2D $O(N)$ model have similarities
  - Non-perturbative scale Λ_{QCD} or m
  - Non-perturbative VEVs $\langle G_{\mu\nu}G^{\mu\nu}\rangle$ or $\langle \alpha \rangle$
  - Infrared divergences
  - Renormalons
- Cured by picking a factorization scale μ between IR and UV

$$\text{IR} \qquad m < \mu < p < M \qquad \text{UV}$$

- Split fields as in Wilsonian renormalization

$$n^i = n_0^i + \varphi^i$$

- Perturbation theory is done with UV fields. IR fields lead to VEVs.

Polyakov-style background field method

- Problems with naive splitting $n^i = n_0^i + \varphi^i$
- Split to maintain $O(N)$ symmetry of IR fields

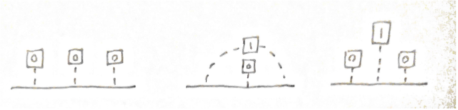
$$n^i = \sigma n_0^i + \sqrt{\lambda}\varphi^a e_a^i$$

- Nth component is always chosen in direction of IR field
- Leads to $SO(N-1)$ gauge invariant action for φ^a
- After dropping terms with unfavorable contractions at $\mathcal{O}(1/N)$

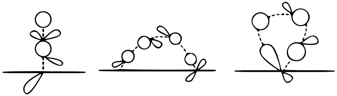
$$S = S_0 + \frac{1}{2}\int d^2x \left[(\partial\varphi)^2 + \left(n_0 \cdot \partial^2 n_0\right)\varphi^2 + \cdots + \mathcal{O}(\lambda)\right]$$

- Similar approach works for SUSY $O(N)$, squashed sphere model

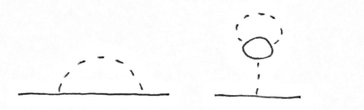
Background field diagrams



- Looks like large N correction diagrams
- Operator VEV part of OPE: Expand base in m^2 and k^2
- Coefficient function part of OPE: Expand α propagator in m^2



BBK approach



- Start with large N corrections to $\langle n(p) \cdot n(-p) \rangle$
- Expand loop integral in powers of m^2 to get OPE

$$\int \frac{d^2k}{(2\pi)^2} \frac{2\pi}{\xi \log\left(\frac{\xi+1}{\xi-1}\right)} \left(\frac{k^2+4m^2}{(p-k)^2+m^2} - 1 \right), \qquad \xi \equiv \sqrt{1+4\frac{m^2}{k^2}}$$

- Is expanding α propagator in integrand directly enough?

Mellin transform

- Problems with interchanging sum and integral

$$\int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2+m^2)^2} =_? \sum_j (-m^2)^j \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^{2(j+1)}} \frac{1}{k^2+m^2}$$

- Instead take Mellin transform

$$\frac{1}{k^2+m^2} = \frac{1}{m^2} \int_C \frac{ds}{2\pi i} \left(\frac{m^2}{k^2}\right)^s \Gamma(s)\Gamma(1-s)$$

- Terms of expansion encoded as poles of $\Gamma(s)\Gamma(1-s)$
- The integration over k introduces new poles. Operator VEVs.

Results of BBK method

- Rewrite α propagator. t will become Borel parameter

$$\frac{1}{\log(\dots)} = \int_0^\infty dt (\dots)^{-t}$$

- Take Mellin transform and integrate over k
- Result is Borel transformed coefficient functions and operators

$$\langle n(-p)n(p)\rangle^{(1)} = \sum_j \int_0^\infty dt \left[e^{-\frac{t}{\lambda(p)}} C_j^{(1)}(p^2, t) m^{2j} + C_j^{(0)}(p^2) \langle O_j \rangle^{(1)}(t) \right]$$

- All renormalon poles on positive t axis cancel. Transseries in λ!

Conclusion

- What did we do?
  - OPE to all orders outside of CFT
  - Transseries to all orders outside of ordinary QM
- Why?
  - The $O(N)$ and related models are toy models for 4D gauge theories
  - Vanishing of VEVs in SUSY theories impacts renormalons?
  - Bring more technical control to resurgence paradigm
- What's next?
  - Genuine super Yang-Mills?
  - Connect operator ambiguities to topological defects in path integral?

Thank you!

- SUSY O(N) model: With Chao-Hsiang Sheu and Misha Shifman: Phys. Rev. D 104, 085016 (2021)
- Squashed sphere model: arXiv:2207.10549 Chap. 5 (2022)