Relativistic Fluids of Topological Defects

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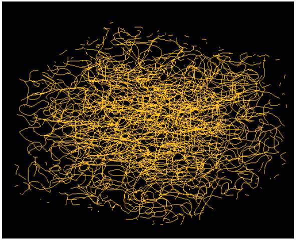
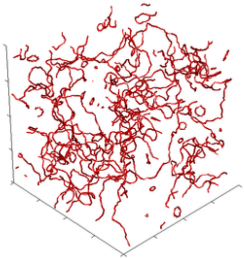
String Networks

- Examples
  - Cosmic Strings
    - ★ Phase transitions
    - ★ Topological defects
  - Quantum Turbulence
  - Magnetic Flux Tubes
- Different scales
- In superfluid, HVBK equations
- Apply idea to other systems
- Outline
  - Relativistic perfect fluids
  - Coarse-grained Nambu-Goto fluid
  - Variational principles
  - Dissipative effects





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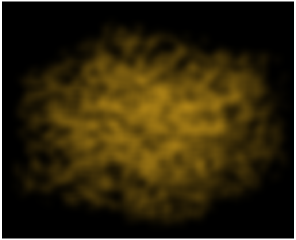
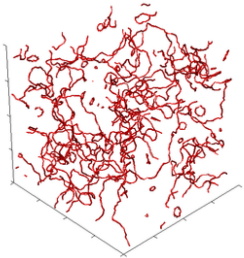




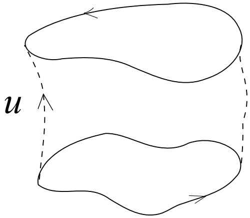
Perfect Fluid

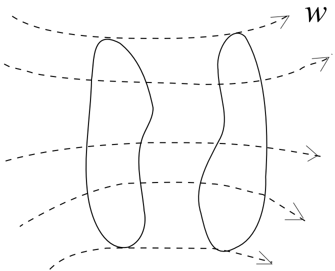
- $T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p g^{\mu\nu}$
- Equation of state: $\rho(n_a)$ is a function of extensive currents
- Locally conserved: $\nabla_\mu(n_a u^\mu) \equiv \nabla_\mu n_a^\mu = 0$
- Chemical potentials (or temperature): $\mu^a \equiv \frac{\partial \rho}{\partial n_a}$
- Pressure via Euler equation: $\rho = -p + \mu^a n_a$
- Legendre transform (n to μ): $n_a = \frac{\partial p}{\partial \mu^a}$

$$
\begin{aligned}
0 = \nabla_\mu T^\mu_{\ \nu} &= \nabla_\mu \left(\mu^a n_a u^\mu u_\nu - p\delta^\mu_\nu\right) \\
&= n_a^\mu \nabla_\mu(\mu^a u_\nu) - n_a \nabla_\nu \mu^a \\
&= 2\, n_a^\mu \nabla_{[\mu}\, \mu^a_{\nu]}
\end{aligned}
$$

Vorticity

- $\omega_{\mu\nu} \equiv 2\,\nabla_{[\mu}\,\mu_{\nu]}$
- Equation of motion: $u^\mu \omega_{\mu\nu} = 0$
- Stokes' theorem $\rightarrow$ circulation
- Closed surface in 4D $\oint \omega = 0$
- So Kelvin circulation theorem
- Spacelike direction $w^\mu \omega_{\mu\nu} = 0$ (Simple bivector)
- Integral describes flux of field lines
- Field lines of w as strings
- So $d\omega = 0$ describes flux conservation





F Tensor

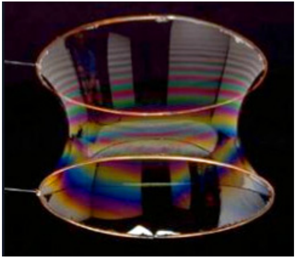
- The dual $F^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\omega_{\rho\sigma}$ is tangent to field lines
- $d\omega = 0$ implies $\nabla_\mu F^{\mu\nu} = 0$
- Similar to conservation of T, n
- $F^{\mu\nu} \equiv \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}$
- $\nabla \cdot \mathbf{B} = 0, \quad \dot{\mathbf{B}} = -\nabla \times \mathbf{E}$
- Still perfect fluid! But also in MHD
- Just as $n^\mu \equiv n u^\mu$ describes charge, $F^{\mu\nu} \equiv \varphi \Sigma^{\mu\nu}$ describes flux
- String fluid: φ itself is thermodynamic quantity

Individual Strings

- Consider 'microscopic' string network
- Worldsheet in spacetime $X^\mu(\tau, \sigma)$
- Velocity and tangent vector

$$U^\mu \equiv X^\mu_{,\tau}$$
$$W^\mu \equiv X^\mu_{,\sigma}$$

- When coarse-grain these become fields
- Nambu-Goto strings
  - Worldlines extremize 'length'
  - Worldsheets extremize spacetime area
- Nambu-Goto Action $-\int d^2\eta \sqrt{-h}$, where $h_{ab} \equiv g_{\mu\nu} X^\mu_{,a} X^\nu_{,b}$



Coarse-Grained Currents

- How do we coarse-grain many individual strings?
- Energy-momentum distributed in spacetime

$$T^{\mu\nu}(x)\sqrt{-g} = \int d^2\eta\sqrt{-h}\; h^{ab}X^{\mu}_{,a}X^{\nu}_{,b}\;\delta(x - X(\eta))$$

- Delta function, so singular
- Add up and coarse-grain in volume $\Delta V(x)$

$$\langle T\rangle^{\mu\nu}(x) = \frac{1}{\Delta V}\int T^{\mu\nu}(x')\sqrt{-g}\, d^4x'$$

- General singular conserved current has form

$$J^{\mu}(x)\sqrt{-g} = \int d^2\eta\, J^a X^{\mu}_{,a}\,\delta(x - X(\eta))$$

- Both spacetime and worldsheet $\nabla_{\mu}J^{\mu} = 0 \iff \partial_a J^a = 0$
- $T^{\mu\nu}$ conservation $\implies \partial_a(\sqrt{-h}h^{ab}X^{\nu}_{,b}) = 0$

F Tensor Again

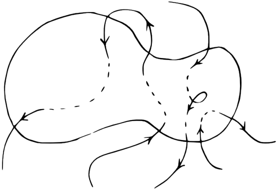
- Currents on worldsheet, doesn't depend on dynamics

$$J^{a\nu} \equiv \epsilon^{ab} X^{\nu}_{,b}, \quad \partial_a J^{a\nu} = 0$$

- Corresponds to an antisymmetric tensor in spacetime

$$F^{\mu\nu}\sqrt{-g} = \int d^2\eta \, J^{a\nu} X^{\mu}_{,a} \, \delta(x - X(\eta))$$

- Coarse-grained $\langle F \rangle$ plays same role as vorticity or EM field tensors
- The conservation $\nabla_\mu F^{\mu\nu} = 0$ related to topological flux of strings



Light-Cone Coordinate Vectors

- A perturbation travels along the string in two directions
- The tangent vectors to these paths in spacetime are denoted A, B

$$A^\mu = U^\mu - W^\mu, \quad B^\mu = U^\mu + W^\mu$$

- $\langle T\rangle^{\mu\nu} = \frac{1}{\Delta V}\int d\rho A^{(\mu}B^{\nu)}, \quad \langle F\rangle^{\mu\nu} = \frac{1}{\Delta V}\int d\rho A^{[\mu}B^{\nu]}$
- Coarse-graining $\sim$ expectation value with respect to energy-density measure
- Statistics of A and B are independent. $\langle T\rangle$ and $\langle F\rangle$ factor
- Kinetic theory of string segments $\implies$ 'local equilibrium' principle

Equations of Motion

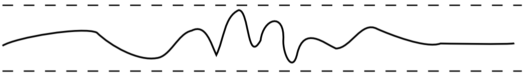
- $\langle T\rangle^{\mu\nu} = \rho \bar{A}^{(\mu}\bar{B}^{\nu)}, \quad \langle F\rangle^{\mu\nu} = \rho \bar{A}^{[\mu}\bar{B}^{\nu]}$
- From $\nabla_\mu \langle T\rangle^{\mu\nu} = \nabla_\mu \langle F\rangle^{\mu\nu} = 0$, in terms of $\bar{U}, \bar{W}$

$$\nabla_\mu(\rho\bar{U}^\mu) = \nabla_\mu(\rho\bar{W}^\mu) = 0$$
$$\bar{U}^\mu\nabla_\mu\bar{U}^\nu - \bar{W}^\mu\nabla_\mu\bar{W}^\nu = 0$$
$$\bar{W}^\mu\nabla_\mu\bar{U}^\nu - \bar{U}^\mu\nabla_\mu\bar{W}^\nu = 0$$

- If $\bar{W} = 0$, dust of small loops. Pressureless.
- Frobenius theorem implies integrable submanifolds.
- Field lines of $\bar{W}$ trace out worldsheets.
- Like macroscopic strings. Nambu-Goto?

Current-Carrying Strings

- Submanifolds are wiggly string worldsheets



- Mass-per-length and tension $MT = \mu_0^2$
- Like energy density, pressure, for a perfect fluid
- From Carter: there is some conserved current ν
- Equation of state $M(\nu) = \mu_0\sqrt{1+\nu^2}$
- Tension defined through $M = T + \mu\nu, \quad \mu \equiv \frac{\partial M}{\partial \nu}$
- More general strings by changing form of $M(\nu)$
- What is the current ν?
- If $\nu = (T_H/\mu_0)s_\nu$, temperature $T \to T_H$

Fluid Lagrangians

- The equation of state $M(\nu)$ like a Lagrangian
- For ordinary perfect fluid, ρ and n

$$\mathcal{L} = -\rho(n), \quad n^2 = -\frac{1}{3!}\tilde{n}^{\lambda\mu\nu}\tilde{n}_{\lambda\mu\nu}$$

- Varying by metric gives correct $T^{\mu\nu}$
- Try same for string fluid: $\rho = \varphi M(\nu)$, where $\varphi^2 = \frac{1}{2}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}$
- Considering units, take $n = \varphi\nu$
- So $\mathcal{L} = -\rho = -\sqrt{(\mu_0\varphi)^2 + (T_H n)^2}$
- Constructing $\mathcal{L}$ from different $M(\nu)$ leads to different submanifolds
- If n dependence vanishes, $\mathcal{L} = -\mu_0\sqrt{\varphi^2}$, is Stachel-Letelier model

Perfect String Fluid

- Begin with any charge and flux $\nabla_\mu n^\mu = \nabla_\mu F^{\mu\nu} = 0$
- Velocity u^μ is unit vector $\frac{1}{n} n^\mu$
- Tangent vector w^μ is unit vector $\frac{1}{\varphi} F^{\mu\nu} u_\nu$
- Given an arbitrary Lagrangian $\mathcal{L}(\varphi^2, n^2)$ we then find

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - (\tau + p) w^\mu w^\nu - p g^{\mu\nu}$$

- Pressure defined so that

$$\rho = -p + mn + \mu\varphi,$$

  with $m = -\mathcal{L}_{,n}$ and $\mu = -\mathcal{L}_{,\varphi}$
- String tension $\tau \equiv \rho - mn$

Magnetohydrodynamics

- Is MHD a 'perfect string fluid'? $\tilde{F}$ is EM field tensor
- Simple bivector condition $u^\mu \tilde{F}_{\mu\nu} = 0$ equivalent to 'frozen-in' magnetic field condition
- Full fluid sum of ordinary fluid and EM parts

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\mathsf{m}} + T^{\mu\nu}_{\mathsf{EM}} \\ &= (\rho + p) u^\mu u^\nu - p g^{\mu\nu} - \tilde{F}^{\mu\rho} \tilde{F}^{\nu}{}_{\rho} + \frac{1}{4} g^{\mu\nu} \tilde{F}^{\rho\sigma} \tilde{F}_{\rho\sigma} \\ &= (\rho + p + \varphi^2) u^\mu u^\nu - \varphi^2 w^\mu w^\nu - (p + \frac{1}{2}\varphi^2) g^{\mu\nu} \end{aligned}$$

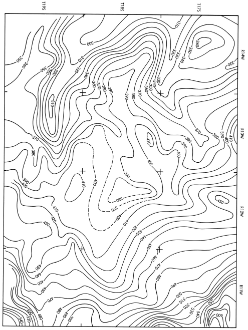
- Check that thermodynamics consistent: $\mathcal{L} = -\rho - \frac{1}{2}\varphi = -\rho_{\mathsf{total}}$,

$$p_{\mathsf{total}} = p + \frac{1}{2}\varphi^2,$$

- Difference between plasma and cosmic string fluid?

2+1d Field

- What are the fields that are varied?
- 2+1d theory: strings in 2D plane
- 'Height' function X describing contour lines
- Construct $F = \star dX \Rightarrow \nabla_\mu F^{\mu\nu} = 0$
- If $\mathcal{L} = -\sqrt{-dX^2}$, contour lines move as Nambu-Goto strings
- Different dependencies on dX^2 lead to pressure
- e.g. Massless scalar field $\mathcal{L} = g^{\mu\nu}\partial_\mu X \partial_\nu X$
- In 3+1d, describes domain walls



Variational Principle

- In 3+1d need two fields X, Y to label submanifolds
- X, Y normalized so that $\tilde{F} \equiv dX \wedge dY$ is flux two-form
- $\nabla_\mu F^{\mu\nu} = 0$ by construction
- To describe fluid particle worldlines need an additional label Z
- Velocity confined to integrable submanifolds by construction
- Normalized so that $\tilde{n} \equiv dX \wedge dY \wedge dZ$ number density volume form
- $\mathcal{L}(\frac{1}{2}\tilde{F}^2, -\frac{1}{3!}\tilde{n}^2)$ varied by $X, Y, Z \ldots$

$$-\frac{3}{2}F^{\lambda\mu}\nabla_{[\kappa}\left(\mu\Sigma_{\lambda\mu]}\right) + 2n^{\lambda}\nabla_{[\kappa}\left(mu_{\lambda]}\right) = 0.$$

- Relabeling symmetry leads to Noether currents. Equations of motion equivalent to conservation. Effective field theory?

Dissipative Fluids Introduction

- Identify a current as entropy: $\rho = -p + Ts + \mu\varphi$
- Perfect string fluid is isentropic

$$u_\nu \nabla_\mu T^{\mu\nu} = T\nabla_\mu(su^\mu) + \mu(\nabla_\mu \varphi u^\mu - \varphi h^\lambda_\mu \nabla_\lambda u^\mu) = 0$$

- In dissipative case there will be entropy production terms. 2nd Law
- Still true $\nabla_\mu T^{\mu\nu} = \nabla_\mu F^{\mu\nu} = \nabla_\mu n^\mu = 0$, no longer perfect fluid form
- No longer a uniquely preferred velocity. Eckart vs Landau-Lifshitz frames
- Alternatively, choose u, w from eigenspace of $F^{\lambda\rho}F_{\rho\mu} \sim h^\lambda_\mu$. String frame vs integrable frame
- If nearly perfect fluid, all frames are close. Principle of frame invariance

Dissipative Terms

- Once we've chosen u, w, we can define T_{eq}

$$T^{\rho\sigma} = T^{\rho\sigma}_{\text{eq}} + 2u^{(\rho}q^{\sigma)} + \dots$$
$$F^{\rho\sigma} = F^{\rho\sigma}_{\text{eq}} + 2w^{[\rho}\nu^{\sigma]} + \dots$$

- Now, from $T^{-1}u_\nu \nabla_\mu T^{\mu\nu} = 0$

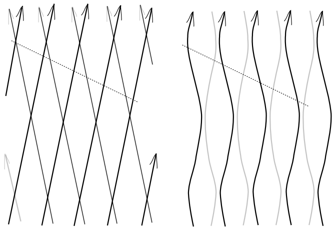
$$\nabla_\mu (su^\mu + \frac{1}{T}q^\mu - \frac{\mu}{T}\nu^\mu) - q^\nu (\nabla_\nu \frac{1}{T} + \frac{1}{T}u^\mu \nabla_\mu u_\nu) + \nu^\nu (\nabla_\nu \frac{\mu}{T} + \frac{\mu}{T} w^\mu \nabla_\mu w_\nu) + \dots = 0$$

- From 1st Law, $ds = \frac{1}{T}d\rho - \frac{\mu}{T}d\varphi$
- From 2nd Law,

$$q^\nu_T = \kappa_T \perp^{\mu\nu} (\nabla_\mu T - T u^\nu \nabla_\nu u_\mu),$$
$$\nu^\mu = \xi_T \perp^{\mu\rho} (\nabla_\rho \frac{\mu}{T} + \frac{\mu}{T} w^\sigma \nabla_\sigma w_\rho)$$

Entropy Production

- There is heat flow and viscosity. What about ν?
- Non-relativistic limit, $\nu = -\xi_T \left(\nabla_\perp \frac{\mu}{T} - \frac{\mu}{T}(\mathbf{w} \cdot \nabla)\mathbf{w} \right)$
- Field lines want to go to lower μ/T
- Second term depends on curl $\mathbf{w} \times \nu = \xi_T \frac{\mu}{T}(\nabla \times \mathbf{w})_\perp$
- Another term with distinct coefficient $\tilde{\mathbf{G}} = \xi_L \frac{\mu}{T}(\nabla \times \mathbf{w})_\parallel$
- Production of loops and wiggles



Resistive Magnetohydrodynamics

- What if we apply the dissipative string fluid idea to MHD?
- In MHD, $\varphi \mathbf{w} = \mathbf{B}$,
- From $\nabla_\mu F^{\mu i} = 0$, Faraday's Law

$$\begin{aligned} \dot{\mathbf{B}} &= -\nabla \times (\mathbf{B} \times \mathbf{v}) - \nabla \times (\mathbf{w} \times \nu) - \nabla \times \tilde{\mathbf{G}} \\ &= -\nabla \times \mathbf{E} \end{aligned} \tag{1}$$

- $\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{\xi}{T}(\nabla \times \mathbf{B}) + \frac{\xi_T}{T^2}\mathbf{B} \times \nabla T$
- By Ampere's Law, $\nabla \times \mathbf{B} \sim \mathbf{J}$. So just Ohm's Law with $\sigma = T\xi^{-1}$
- Temperature gradient term, Nernst effect

Hydrostatic Equilibrium

- What if we have a dissipative system in equilibrium?
- Setting all dissipative terms to zero constrains system
- Can prove $u^\mu \nabla_\mu f(s, T, \varphi, \dots) = 0$
- Timelike Killing vector $\nabla_{(\mu} \frac{1}{T} u_{\nu)} = 0$
- Curvature in strings balanced by gradients of μ

$$\nu = 0 \implies (\mathbf{w} \cdot \nabla)\mathbf{w} = \nabla_\perp \ln \frac{\mu}{T}$$

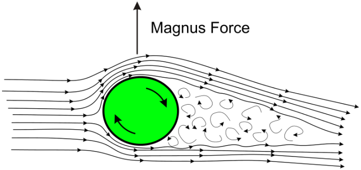
- Spacelike irrotational vector $\nabla_{[\mu} \frac{\mu}{T} w_{\nu]} = 0$
- Natural coordinate system
- Astrophysical applications?

Thank you!

- Any questions?
- References and more info in papers:
  - Field Theory for Perfect String Fluids: arXiv:1410.5843 [hep-th]
  - Dissipative String Fluids: arXiv:1412.3135 [hep-th]
  - String Fluid in Local Equilibrium: arXiv:1406.1226 [hep-th]

Relabeling Symmetries

- X, Y, Z only appear in combinations $\tilde{F}, \tilde{n}$
- $Z \to Z + F(X, Y)$ symmetry. Carter's dual current
- Symmetries of X, Y, symplectic transformations
- Effective field theory picture (Dubovsky et al), higher derivative terms
- Instead of dependence on $\tilde{n}$, $mu^\mu = dZ' + XdY$ leads to superfluid description
- $dX \wedge dY \wedge dZ'$ also allowed by symmetry, describes Zhukovsky lift (Carter-Langlois model)



- Dissipation by integrating out irrelevant degrees of freedom

Higher-Order Effects

- Longitudinal heat flow $Q_L = \kappa_L w^\mu(\nabla_\mu T - T u^\nu \nabla_\nu u_\mu)$
- Appearance of heat equation $\dot{T} = \frac{\kappa_L}{C}\partial_w^2 T$
- But entropy current s is a function of T, F expanded about an equilibrium state
- Including second-order terms, e.g. $-\frac{1}{2}k u^\mu Q_L^2$

$$kCT^2\ddot{T} + \frac{C}{\kappa_L}\dot{T} = \partial_w^2 T$$

- s should not depend on the frame it is expanded about. This restricts form of second-order terms
- Q_L transforms differently from other quantities, so second-order term is fixed
- Speed of second sound, $c_s = \sqrt{\frac{\tau}{\rho}} = \sqrt{1 - \left(\frac{T}{T_H}\right)^2}$